## Game Theory <br> January 2019 exam

Name:
Group:
You have two and a half hours to complete the exam. No calculators allowed.

## I. Short questions (5 points each)

I. 1 In a simultaneous game with two players and two strategies each, which is the maximum number of Nash equilibria that can be found in pure strategies? What is the minimum number? Provide an example of each case.

Maximum number: 4, it is the maximum number of strategy combinations and there are games with that number of NEa. Minimum number: 0 .

| 1,1 | 1,1 |
| :--- | :--- |
| 1,1 | 1,1 |
| 4 NEa |  |


| $1,-1$ | $-1,1$ |
| :---: | :---: |
| $-1,1$ | $1,-1$ |
| 0 NEa |  |

I. 2 If two nodes in the extensive form game belong to the same information set, then each one of these two nodes will have the same number of branches after them. True or false?

True. If they have different number of branches after them, the player can tell them apart.
I. 3 What is the difference between first and second price auctions?

In the first price auction the winner pays her bid. In the second price auction, the winner pays the second highest bid.
I. 4 Consider an election by majority with three voters (1, 2 and 3 ) and three alternatives (A, B and $C$ ). In case of a tie, $C$ is elected. Voter 1 rank the alternatives in the order $A B C$, Voter 2 in the order BAC, and 3 in the order CAB. Find all Nash equilibria in pure strategies satisfying (i) A is elected, and (ii) no voter votes for her least preferred alternative.

There are only 4 possible candidates that satisfy (i) and (ii):
(A, A, A): It is a NE. No individual deviation changes the result.
(A, A, C): It is a NE. If Voter 1 deviates, $A$ is not elected and he will be worse off. If Voter 2 deviates C will be elected and he will be worse off. If Voter 3 deviates nothing happens.
(A, B, A): Not a NE. If Voter 3 deviates to $C, C$ will be elected and he will be better off.
(B, A, A): Not a NE. If Voter 2 deviates to B, B will be elected and she will be better off.

## II. Problems (20 points each)

II. 1 The 10 neighbors in a small city share a park that they all prefer to keep as clean as possible, but they all find it costly to use the trash bins. A clean park is valued at 100 by each neighbor, and from there they subtract 10 for each neighbor that decides not to clean after them. The cost of being clean is 15 . If a neighbor decides not to be clean, she will suffer a moral disapproval equal to twice the number of neighbors that chooses to be clean. This moral disapproval is a cost to subtract from the valuation.
(a) Define the above situation as a normal form game. Hint: to define the utility functions just consider the one by Neighbor $i$ when, among all other neighbors, there are $k$ that decided to be clean (and, thus, $9-k$ decided not to be clean). (5 points)
(b) Assume now that all neighbors decide to be clean. Is this situation a NE? (5 points)
(c) Assume now that all neighbors decide not to be clean. Is this situation a NE? (5 points)
(d) Assume now that 5 neighbors decide to be clean and 5 decide against that. Is this situation a NE? (5 points)
(a) $u_{i}(i$ clean, $k$ others clean, $9-k$ others unclean $)=100-10(9-k)-15=10 k-5$. $u_{i}(i$ unclean, $k$ others clean, $9-k$ others unclean $)=100-10(10-k)-2 k=8 k$.
(b) $u_{i}($ all clean $)=100-15=85 . u_{i}($ all clean but $i)=72$. Neighbor $i$ does not deviate. The situation is a NE.
(c) $u_{i}($ all unclean $)=0 \cdot u_{i}($ all unclean but $i)=-5$. Neighbor $i$ does not deviate. The situation is a NE.
(d) $u_{i}(i$ clean, 4 others clean, 5 others unclean $)=35$.
$u_{i}(i$ unclean, 4 others clean, 5 others unclean $)=32$.
A clean neighbor does not deviate.
$u_{i}(i$ unclean, 5 others clean, 4 others unclean $)=40$.
$u_{i}(i$ clean, 5 others clean, 4 others unclean $)=45$.
An unclean neighbor deviates.
It is not a NE.
II. 2 Each of two neighbors have a free day that they can use to spend with their families or to do some community work in the neighborhood. If $t_{i}$ is the amount of time that Neighbor $i$ dedicates to community work, then $\left(1-t_{i}\right)$ is the time she spends with the family $\left(t_{i} \in[0,1]\right)$. They both like a working community and to spend time with the family. Their respective utility functions are:

$$
\begin{aligned}
& u_{1}\left(t_{1}, t_{2}\right)=2 t_{1}-t_{1} t_{2}-\frac{1}{2} t_{1}^{2}+\left(1-t_{1}\right) \\
& u_{2}\left(t_{1}, t_{2}\right)=2 t_{2}-t_{1} t_{2}-\frac{1}{2} t_{2}^{2}+\left(1-t_{2}\right),
\end{aligned}
$$

where $2 t_{i}-t_{i} t_{j}-\frac{1}{2} t_{i}^{2}$ represents Neighbor $i$ 's utility of living in a working community, and $\left(1-t_{i}\right)$ is the utility of spending time with the family. The process to decide about the use of time is sequential. First, Neighbor 1 decides and, once Neighbor 2 observes the decision by Neighbor 1, Neighbor 2 makes her own.
(a) Find the subgame perfect Nash equilibrium. (8 points)
(b) Find the equilibrium path and the neighbors' utilities in the equilibrium. (6 points)
(c) If the decisions are made simultaneously, will Neighbor 1 be better off relative to the sequential case? ( 6 points)
(a) Neighbor 2's best reply:
$\max _{t_{2}} u_{2}\left(t_{1}, t_{2}\right)=2 t_{2}-t_{1} t_{2}-\frac{1}{2} t_{2}^{2}+\left(1-t_{2}\right)$,
First order conditions for a maximum are $2-t_{1}-t_{2}-1=0$, giving $t_{2}=1-t_{1}$. Notice the second order conditions for a maximum are satisfied: $-1<0$.

Anticipating this reaction Neighbor 1 solves:
$\max _{t_{1}} u_{1}\left(t_{1}, t_{2}\right)=2 t_{1}-t_{1} t_{2}-\frac{1}{2} t_{1}^{2}+\left(1-t_{1}\right)$
s.t. $t_{2}=1-t_{1}$
or
$\max _{t_{1}} 2 t_{1}-t_{1}\left(1-t_{1}\right)-\frac{1}{2} t_{1}^{2}+\left(1-t_{1}\right)=\frac{1}{2}\left(t_{1}\right)^{2}+1$,
This function is increasing in $t_{1}$, what means it has no global or local maximum, and the maximum is the highest possible value for $t_{1}$. I.e., $t_{1}=1$.
Alternatively, one can check that S.O.C. give: $\frac{1}{2}>0$, which implies there is no local or global maximum.

The SPNE is $\left(t_{1}=1, t_{2}=1-t_{1}\right)$.
(b) Equilibrium path: $\left(t_{1}=1, t_{2}=0\right)$. Equilibrium utilities: $u_{1}=\frac{3}{2}, u_{2}=1$.
(c) Best replies are $t_{2}=1-t_{1}$ and $t_{1}=1-t_{2}$. They are the same equation that define infinitely equilibria: all pairs $\left(t_{1}, t_{2}\right)$ satisfying $t_{1}+t_{2}=1$. In (a) we saw that the best case for Neighbor 1 is when $t_{1}=1$ (and $t_{2}=0$ ), which means that, among all these equilibria, only one gives the same utility, while all others give less. There is no advantage for Neighbor 1 to be in the simultaneous case.
II. 3 Consider the repeated game which consists in playing the following stage game infinitely many times with discount factor $\delta=\frac{3}{4}$.

Player 2

Player 1

|  | C |
| :---: | :---: |
| D |  |
| C | 4,4 |
| D | 0,6 |
|  | 6,0 |
|  |  |

The tit-for-tat strategy for Player 1 is defined as follows: at $t=1$ play C ; at $t>1$ play whatever Player 2 played at $t-1$, and similarly for Player 2. Consider the following deviations from tit-fortat by Player 1 at the beginning of the game:

Deviation 1: In the first period play D , and then follow tit-for-tat.
Deviation 2: In the first two periods play D and then follow tit-for-tat.
(a) Show that no one of these deviations gives the deviator more than tit-for-tat. (10 points)
(b) Show that no other deviation from the first period gives the deviator more than the best of deviations 1 and 2. In other words, tit-for-tat is indeed a Nash equilibrium of the game. (4 points)

Notice that tit-for-tat makes different recommendations to play in four different types of subgames at $t>1$ : those starting after the previous play was $(\mathrm{C}, \mathrm{C}),(\mathrm{C}, \mathrm{D}),(\mathrm{D}, \mathrm{C})$ or $(\mathrm{D}, \mathrm{D})$.
(c) Show that in at least one of those subgames, tit-for-tat is not a Nash equilibrium (and, then, is not a SPNE in the whole game). (6 points)
(a) By following tit-for-tat, players get 4 in each period, what gives $u_{i}=\frac{4}{1-\frac{3}{4}}=16$.

Deviation 1:
If a player uses the deviation, she gets 6 in the first period and, then, 0 and 6 alternating, for a utility of $u_{i}=6+0 \delta+6 \delta^{2}+0 \delta^{3}+6 \delta^{4}+0 \delta^{5}+\cdots=6+6 \delta^{2}+6 \delta^{4}+\cdots=\frac{6}{1-\delta^{2}}=\frac{6}{1-\left(\frac{3}{4}\right)^{2}}=\frac{96}{7}$, which is smaller than 16 . Hence no player wants to use Deviation 1.

Deviation 2:
If a player uses the deviation, she gets 6 in the first period and 2 from there on, for a utility of $u_{i}=$ $6+2 \delta+2 \delta^{2}+2 \delta^{3}+\cdots=6+\frac{2 \frac{3}{4}}{1-\frac{3}{4}}=12$, which is smaller than 16 . Hence no player wants to use Deviation 2.
(b) Tit-for-tat implies playing: (C,C), (C,C), (C,C), (C,C), (C,C), (C,C), ...

Deviation 1 by Player 1 implies playing: (D,C)*, (C,D), (D, C), (C,D), (D,C), (C,D),...
Deviation 2 by Player 1 implies playing: (D,C)*, (D,D)*, (D,D), (D,D), (D,D), (D,D), ..
${ }^{*}$ ) indicates the period of the deviation.
If Player 1 deviates in periods 1 and 3 , the play will be: $(\mathrm{D}, \mathrm{C})^{*},(\mathrm{C}, \mathrm{D}),(\mathrm{C}, \mathrm{C})^{*},(\mathrm{C}, \mathrm{C}),(\mathrm{C}, \mathrm{C})$, (C,C),...
If Player 1 deviates in periods 1 and 4 , the play will be: $(D, C)^{*},(C, D),(D, C),(D, D)^{*},(D, D)$, (D,D),...

We observe that deviations in two periods imply one of the three possibilities: (i) alternating (C,D) and (D,C) forever, (ii) (C,C) forever, or (iii) (D,D) forever. A deviation in three or more periods does the same thing. We already know that alternating and (D,D) forever are both a punishment big enough to deter the temptation of 6 in the first period. What about the other
possibility of deviating in the first period to go back to (C,C) two periods later? The only difference is in the first two periods, and $6+0 \frac{3}{4}<4+4 \frac{3}{4}$. It does not pay either.
(c) Consider subgame after (C, D), according to tit-for-tat players alternate between (D, C) and (C, D). Player 1 gets payoffs alternating 6 and 0 , with a utility of $u_{1}=6+0 \delta+6 \delta^{2}+0 \delta^{3}+$ $\cdots=\frac{96}{7}$. If she deviates and plays C instead of D , she gets 4 in all periods, with $u_{1}=16$.
II. 4 Consider a Cournot duopoly with two firms operating in a market where the inverse demand function is $P=A-Q$, and $Q=q_{1}+q_{2}$ is the total output in the market. Suppose that there is uncertainty about the value of $A$, which with probability $p=\frac{1}{2}$ takes the value $A=36$, and with probability $1-p=\frac{1}{2}$ is $A=24$. The production costs are zero for both firms. Firm 2 is informed about the value of $A$, but Firm 1 only knows the above probabilities. Both firms decide independently and simultaneously their respective output, $q_{1}$ and $q_{2}$.
(a) Describe the above situation as a Bayesian game. (5 points)
(b) Compute the Bayesian equilibrium. ( 9 points)

Suppose now that Firm 2 can credibly inform Firm 1 about the value of $A$. (E.g., it can let Firm 2 visit its premises.)
(c) Will Firm 2 give the information if $A=36$ ? And if $A=24$ ? What would Firm 1 infer if Firm 2 does not communicate the value of $A$ ? ( 6 points)
(a) $N=\{1,2\}$;
$T_{1}=\left\{t_{1}^{1}\right\}, T_{2}=\left\{t_{2}^{1}, t_{2}^{2}\right\} ;$
$\left(p\left(t_{2}^{1} \mid t_{1}^{1}\right)=\frac{1}{2}, p\left(t_{2}^{2} \mid t_{1}^{1}\right)=\frac{1}{2}\right),\left(p\left(t_{1}^{1} \mid t_{2}^{1}\right)=1\right),\left(p\left(t_{1}^{1} \mid t_{2}^{2}\right)=1\right) ;$
$A_{t_{1}^{1}}=A_{t_{2}^{1}}=A_{t_{2}^{2}}=\left\{q_{t} \in[0, \infty)\right\} ;$
$u_{t_{1}^{1}}=\frac{1}{2}\left(36-q_{t_{1}^{1}}-q_{t_{2}^{1}}\right) q_{t_{1}^{1}}+\frac{1}{2}\left(24-q_{t_{1}^{1}}-q_{t_{2}^{2}}\right) q_{t_{1}^{1}}$,
$u_{t_{2}^{1}}=\frac{1}{2}\left(36-q_{t_{1}^{1}}-q_{t_{2}^{1}}\right) q_{t_{2}^{1}}$,
$u_{t_{2}^{2}}=\frac{1}{2}\left(24-q_{t_{1}^{1}}-q_{t_{2}^{2}}\right) q_{t_{2}^{2}}$.
(b) Respective FOC for types give (check SOC for a maximum are satisfied):
$q_{t_{1}^{1}}=\frac{1}{2} \frac{36-q_{t_{2}^{1}}}{2}+\frac{1}{2} \frac{24-q_{t_{2}^{2}}}{2}, q_{t_{2}^{1}}=\frac{36-q_{t_{1}^{1}}}{2}, q_{t_{2}^{2}}=\frac{24-q_{t_{1}^{1}}}{2}$.
Solving the system, one gets the Bayesian equilibrium: $\left(q_{t_{1}^{1}}=10, q_{t_{2}^{1}}=13, q_{t_{2}^{2}}=7\right)$.
(c) In the equilibrium profits are $u_{t_{1}^{1}}=100, u_{t_{2}^{1}}=169, u_{t_{2}^{2}}=49$.
(i) If $t_{2}^{1}$ informs credibly about $A$, then $t_{1}^{1}$ reaction function will be $q_{t_{1}^{1}}=\frac{36-q_{t_{2}^{1}}}{2}$, which, together with $q_{t_{2}^{1}}=\frac{36-q_{t_{1}^{1}}}{2}$ gives the solution $q_{t_{1}^{1}}=12, q_{t_{2}^{1}}=12$, and profits for $t_{2}^{1}: u_{t_{2}^{1}}=$ 144, lower than 169, so it will not inform about $A$.
(ii) If $t_{2}^{2}$ informs credibly about $A$, then $t_{1}^{1}$ reaction function will be $q_{t_{1}^{1}}=\frac{24-q_{t_{2}^{2}}}{2}$, which, together with $q_{t_{2}^{2}}=\frac{24-q_{t_{1}^{1}}}{2}$, gives the solution $q_{t_{1}^{1}}=8, q_{t_{2}^{2}}=8$, and profits for $t_{2}^{2}: u_{t_{2}^{2}}=64$, higher than 49 , so it will inform about $A$.
(iii) If $t_{2}^{1}$ is not informed about $A$ by Firm 2, it will infer that Firm 2 is of type $t_{2}^{1}$. If it were of type $t_{2}^{2}$, Firm 2 would have informed about $A$.

