Game Theory
June Exam 2016

Name: **Group:**

You have two and a half hours to complete the exam.

I. Short questions (5 points each)

I.1 Show an example of a normal form game with infinitely many Nash equilibria.

1, 1	1, 1
1, 1	1, 1

I.2 Consider the results we observed when we studied negotiations (bargaining) with dynamic games: Is there an advantage in making the last offer? Is there an advantage in being more patient? Explain in few lines why or why not.

The player making the last offer can make a take-it-or-leave-it offer to the other player, and be at an advantage. This advantage is translated to previous periods, although it decreases because of the discount factor.

Being more patient means that more offers can be rejected today in exchange for a better deal tomorrow. This puts the patient player in a better position.

I.3 Show a static game with two Nash equilibria in pure strategies such that, when it is repeated twice, there exists at least one subgame perfect Nash equilibrium in which players obtain payoffs larger than those in either of the two equilibria in some of the repetitions.

	A	B	C
A	5, 5	0, 6	0, 0
B	6, 0	3, 3	0, 0
C	0, 0	0, 0	1, 1

(B, B) and (C, C) are both NE. Play (A, A) in the first period, play (B, B) in the second period if (A, A) was played in the first period, otherwise play (C, C) in the second period.

I.4 In a Bayesian game of two players, the set of actions for Player 1 is $S_1 = \{a, b\}$ and the set of actions for Player 2 is $S_2 = \{x, y, z\}$. The types for Player 1 are $T_1 = \{h_1, h_2\}$ and the types for Player 2 are $T_2 = \{l_1, l_2, l_3\}$. Which are the conditional probabilities that one needs to describe to complete the definition of the game?

For h_1: $(prob(l_1|h_1), prob(l_2|h_1), prob(l_3|h_1))$.
For h_2: $(prob(l_1|h_2), prob(l_2|h_2), prob(l_3|h_2))$.
For l_1: $(prob(h_1|l_1), prob(h_2|l_1)$.
For l_2: $(prob(h_1|l_2), prob(h_2|l_2)$.
For l_3: $(prob(h_1|l_3), prob(h_2|l_3)$.

II. Problems

II.1 (25 points) Three firms compete in a market. They must decide simultaneously the quantities q_1, q_2 and q_3 of the product to sell in the market. The market price at which the units will be sold are given by the inverse demand function $P(q_1, q_2, q_3) = 72 - q_1 - q_2 - q_3$. All firms have zero costs.

(a) Compute the best reply functions for each of the firms. (5 points)
(b) Compute the Nash-Cournot equilibrium, for the firms. Compute also their profits in the equilibrium. (5 points)

Suppose now that Firm 2 is considering to buy Firm 3, and that if it does, then Firms 2 and 3 behave as a single firm.

(c) Compute the new equilibrium. (5 points)
(d) How much is Firm 2 willing to pay to buy Firm 3? Will Firm 3 accept the offer? (5 points)
(e) Both Firm 1 and the consumers can ask the market regulator to prevent the buy. Will Firm 1 be willing to prevent Firm 2 from buying Firm 3? What about consumers? (5 points)

(a) Firm *i* solves:
$\mathbf{max}_{q_i}(72 - q_1 - q_2 - q_3)q_i$, which implies $q_i = \frac{72-q_j-q_k}{2}$.

(b) By symmetry: $q_i = q_j = q_k$, and then $q_i = \frac{72}{4} = 18$ for $i = 1, 2, 3$. $p = 72 - 3\times18 = 18$, $\Pi_i = 18\times18 = 324$.

(c) Firm *i* solves:
$\mathbf{max}_{q_i}(72 - q_1 - q_{23})q_i$, which implies $q_i = \frac{72-q_j}{2}$. By symmetry: $q_i = q_j$, and then $q_i = \frac{72}{3} = 24$ for $i = 1, 23$.

(d) $p = 72 - 2\times24 = 24$, $\Pi_i = 24\times24 = 576$. Firm 2 wins $576 - 324 = 252$, which is the most it is willing to pay to buy Firm 3. Firm 3 needs to be paid at least 324 to accept, it will not accept.

(e) Firm 1 is making 576 in the new situation. It will not complain. Price is higher (and total quantity smaller), so the consumers will complain. Consumer surplus is $CS' = \frac{(72-24)\times48}{2} = 1152$ versus $CS = \frac{(72-18)\times54}{2} = 1458$.

II.2 (25 points) Two firms, Peurot and Fol, compete in the car market, where demand is given by $P(Q) = 2 - Q$ and technology is such that the production marginal cost is $c = 1$. In this market, Peurot is the leader, (it chooses first) and Fol is the follower, that chooses its quantity knowing the quantity chosen by Peurot. Fol cares not only about profits (π_F) but on sales as well (q_F), as it wants to get a market share. In particular, the utility function of Fol is

$$U_F(q_p, q_F) = \alpha\, \pi_F(q_P, q_F) + (1-\alpha)q_F$$

while the leader only cares for profits,

$$U_P(q_{P,} q_F) = \pi_P(q_{P,} q_F).$$

(a) If $\alpha > 1/2$ compute the subgame perfect Nash equilibrium. Find for what values of α the leader produces more than the follower in the equilibrium? (15 points)

(b) Suppose now that $\alpha = \frac{1}{4}$. Compute the subgame perfect Nash equilibrium. (10 points)

(a) Fol moves last, and will solve:
$\max_{q_F} U_F(q_p, q_F) = \alpha(2 - q_P - q_F - 1)q_F + (1-\alpha)q_F$
First order conditions give $q_F = \frac{1-\alpha q_P}{2\alpha}$.
In the first period Peurot solves:
$\max_{q_P} U_P(q_p) = \left(2 - q_P - \frac{1-\alpha q_P}{2\alpha} - 1\right) q_P$
First order conditions give $q_P = 1 - \frac{1}{2\alpha}$, and then $q_F = \frac{3}{4\alpha} - \frac{1}{2}$.
The SPNE is $q_P = 1 - \frac{1}{2\alpha}$, $q_F = \frac{1-\alpha q_P}{2\alpha}$ if $q_P < \frac{1}{\alpha}$, $q_F = 0$ if $q_P \geq \frac{1}{\alpha}$.
To get $q_P > q_F$ we need $\frac{3}{4\alpha} - \frac{1}{2} > 1 - \frac{1}{2\alpha}$, or $\alpha > \frac{5}{6}$.

(b) Substituting: $q_P\left(\alpha = \frac{1}{4}\right) = 1 - \frac{1}{2\times\frac{1}{4}} = -1 < 0$. Because quantities cannot be negative, $q_P = 0$, with $q_F = \frac{1-\alpha q_P}{2\alpha} = \frac{1}{2\times\frac{1}{4}} = 2$. The SPNE is $q_P = 0$, $q_F = \frac{1-\frac{1}{4}q_P}{2\frac{1}{4}} = 2 - \frac{1}{2}q_P$ if $q_P < 4$, $q_F = 0$ if $q_P \geq 4$.

II.3 (15 points) Consider the following stage game in which each player chooses between C and D, repeated multiple times:

		Player 2	
		C	D
Player 1	C	3, 3	0, 4
	D	0, 1	2, 0

Suppose that each player discounts her payoffs with a discount factor $\delta = 0.9$.

(a) Suppose the game is repeated two times. Find all the subgame perfect Nash equilibria. (5 points)
(b) Suppose the game is repeated infinitely. Show a SPNE in which (C,C) is played along the equilibrium path. (10 points)

(a) There is no equilibrium in pure strategies. To find the equilibria in mixed strategies write $p = prob_1(C)$ for Player 1 and $q = prob_2(C)$ for Player 2. $Eu_1(C) = Eu_1(D)$: $3q = 2(1-q)$, which gives $q = 2/5$. Similarly $Eu_2(C) = Eu_2(D)$: $3p + (1-p) = 4p$, which gives: $p = 1/2$.
The only NE is $(\frac{1}{2}[C] + \frac{1}{2}[D],\ \frac{2}{5}[C] + \frac{3}{5}[D])$.
Being that the only NE, in the second period, that is the NE in all subgames.
Being that the equilibrium in the second period is not affected by the play in the first period, the NE will also be played in the first period.
The SPNE is $(\frac{1}{2}[C] + \frac{1}{2}[D],\ \frac{2}{5}[C] + \frac{3}{5}[D])$ in period 1, and $(\frac{1}{2}[C] + \frac{1}{2}[D],\ \frac{2}{5}[C] + \frac{3}{5}[D])$ in period 2 in all subgames, which gives utilities $(\frac{6}{5}, 2)$.

(b) Try the trigger strategy *T*: play (C, C) in the first period and then (C, C) in period *t* as long as (C, C) has been played for all periods before *t*, and play $(\frac{1}{2}[C] + \frac{1}{2}[D],\ \frac{2}{5}[C] + \frac{3}{5}[D])$ otherwise.

(i) It is a NE: $u_2(T) = 3 \times \frac{1}{1-0.9} = 30$, while the utility of the best deviation (*Dev*) is $u_2(Dev_2, T_1) = 4 + 2 \times \frac{0.9}{1-0.9} = 22$. (With trigger strategies we only need to check one-period devations.) Player 1 will get 0 rather than 3 if she deviates in the first period: no need to check more.

(ii) It is a NE in every subgame after (C, C) in all previous periods: in these subgames, the game is exactly as the whole game and the trigger strategy specifies the same strategy choice as in the whole game, which means that if the trigger strategy was a NE of the whole game it is also a NE in these subgames.

(iii) It is a NE in every subgame after no (C, C) in some previous perido: in these subgames the trigger strategy specifies the playing of the one-stage NE in every subsequent period, which is a NE of the repeated game.

Because it is a NE in every subgame, the trigger strategy is a SPNE.

II.4 (15 points) Consider the following two games, Game I and Game II. With probability p, the payoffs of the two players, A and B, are those of Game I and with probability $1 - p$ the payoffs are those of Game II. Player A is informed which of the two games is chosen, but player B does not know which of the two games is being played. Both players have to decide simultaneously: A chooses between *x* and *y*, and B chooses between *m* and *n*.

Game 1
Probability *p*

	m	n
x	1, 1	0, 0
y	0, 0	1, 1

Game II
Probability 1-*p*

	m	n
x	0, 0	0, 0
y	1, 1	1, 1

(a) Write down all the elements of the Bayesian game. (3 points)
(b) Calculate the pure-strategy Bayesian Nash equilibria of this game. (12 points)

(a) Players = $\{A, B\}$.
Types: Types of Player $A = \{t_A^1, t_A^2\}$, types of Player $B = \{t_B\}$
Beliefs: $(prob(t_B|t_A^1) = 1))$, $(prob(t_B|t_A^2) = 1)$, $(prob(t_A^1|t_B) = p, prob(t_A^2|t_B) = 1 - p)$.
Actions: for types of Player $A = \{x, y\}$, for types of Player $B = \{m, n\}$,
Strategies: for $A = \{(x, x), (x, y), (y, x), (y, y)\}$, for Player $B = \{m, n\}$.

(b) *A* will chose *y* in Game II as it strictly dominates *x*.
Is there any equilibrium with *A* choosing *x* in Game I? If this is the case, *B*'s best reply is to choose *m* (she gets $p\times 1 + (1 - p)\times 1 = 1$, rather than $p\times 0 + (1 - p)\times 1 = (1 - q)$ if she chooses *n*). If *B* chooses m, the best reply for A is to choose *x* in Game I and *y* in Game II. Then $((x, y), m)$ is a Bayesian Nash equilibrium. Similar arguments show that $((y, y), n)$ is also a BNE.