Game Theory
Exam June 2014

Name:
Group:
You have two and a half hours to complete the exam.

## I. Short questions (20 points).

I. 1 Provide an example in which the Nash equilibrium contains weakly dominated strategies. Could you give a similar example with strongly dominated strategies?
I. 2 Is backward induction a special case of subgame perfection? Or is it the other way around?
I. 3 Let $s$ and $s$ ' be two Nash equilibria in a static game $G$. If $G$ is repeated $T$ times, to play $s$ the pair periods and $s$ ' the odd ones is a subgame perfect equilibrium. True or false?
I. 4 Define the elements of a Bayesian game.

## II. Problems. You must answer four of the following five problems. (20 points each).

II. 1 Two students of Carlos III University (Ana and Juan) have to study for an exam, where the student with the highest grade will receive a $100 €$ prize. The second gets nothing. Player $i$ ( $i=$ Ana, Juan) has to choose a level of effort $e_{i}$, measured in hours of study. Assume that the probability $p_{i}$ of Player $i$ getting the prize is equal to the proportion of the own effort over the total effort by both players. The cost of one hour of effort is $10 €$. If both students maximize expected profits, given by $G_{i}=100 p_{i}-10 e_{i}$ for Player $i$, compute the effort level in the Nash equilibrium of the game.
$\max _{e_{i}} G_{i}=100 \frac{e_{i}}{e_{i}+e_{j}}-10 e_{i}$
FOC: $100 \frac{e_{j}}{\left(e_{i}+e_{j}\right)^{2}}-10=0$
For $j$ we obtain a similar expression.
By symmetry: $e_{i}=e_{j}$, and we get: $e_{i}=e_{j}=2,5$.
SOC is: $-\left(e_{i}+e_{j}\right)<0$, which means that the FOC are the conditions for a maximum.
II. 2 Consider the game depicted next.
(b) Find the subgame perfect equilibria. (4 points)


Suppose now that Player 1 does not know whether Player 2 has chosen $a$ or $b$.
(c) Represent the new game graphically. Which is the strategy set for each player? (4 points)
(d) Compute the Nash equilibria of the subgame starting when Player 2 must choose between $a$ and $b$. ( 6 points)
(e) Find the subgame perfect equilibria of the whole game. (6 points)
(a) $\mathrm{SPNE}=\{((\mathrm{s}, \mathrm{c}, \mathrm{f}),(\mathrm{b}))\}$
(b)


Strategies for $1=\{(\mathrm{t}, \mathrm{c}),(\mathrm{t}, \mathrm{d}),(\mathrm{s}, \mathrm{c}),(\mathrm{s}, \mathrm{d})\}$
Strategies for $2=\{a, b\}$
(c) $\mathrm{NE}=\left\{(\mathrm{a}, \mathrm{c}),(\mathrm{b}, \mathrm{d}),\left(\left(\frac{3}{5}[\mathrm{a}]+\frac{2}{5}[\mathrm{~b}]\right),\left(\frac{4}{5}[\mathrm{c}]+\frac{1}{5}[\mathrm{~d}]\right)\right)\right\}$
(d) $\operatorname{SPE}=\left\{(\mathrm{t},(\mathrm{a}, \mathrm{c})),(\mathrm{s},(\mathrm{b}, \mathrm{d})),\left(\mathrm{t},\left(\left(\frac{3}{5}[\mathrm{a}]+\frac{2}{5}[\mathrm{~b}]\right),\left(\frac{4}{5}[\mathrm{c}]+\frac{1}{5}[\mathrm{~d}]\right)\right)\right)\right\}$
II. 3 Two firms compete in a differentiated goods market by selecting their prices. Firm $i$ 's demand is given by $q_{i}\left(p_{i}, p_{j}\right)=20-\frac{4}{3} p_{i}+\frac{2}{3} p_{j}, i, j=1,2, i \neq j$, and its cost is given by $C\left(q_{i}\right)=5 q_{i}$. Suppose first that the two firms choose their prices simultaneously.
(a) Find the firms' reaction functions and show them in a graph. Find the firms' prices and profits in the unique Nash equilibrium. Find also the profits in the equilibrium. (8 points)

Suppose now that Firm 1 chooses its price first and then Firm 2, after observing the price of its rival, chooses its own price.
(b) Represent the new game in an extensive form. Indicate the information sets of each firm. (4 points)
(c) Find the subgame perfect Nash equilibrium prices and profits. (8 points)
(a) $\max _{p_{i}}\left(p_{i}-5\right)\left(20-\frac{4}{3} p_{i}+\frac{2}{3} p_{j}\right)$

CPO: $\left(20-\frac{4}{3} p_{i}+\frac{2}{3} p_{j}\right)+\left(p_{i}-5\right)\left(-\frac{4}{3}\right)=0$,
Operating we find the reaction function for $i$ : $p_{i}=\frac{40+p_{j}}{4}$.
Similarly for $j$.
By symmetry: $p_{i}=p_{j}=\frac{40}{3}$,
from there: $\mathrm{NE}=\left\{\left(\frac{40}{3}, \frac{40}{3}\right)\right\}$.
(c) $\max _{p_{1}}\left(p_{1}-5\right)\left(20-\frac{4}{3} p_{1}+\frac{2}{3} p_{2}\right)$
s.a: $p_{2}=\frac{40+p_{1}}{4}$
i.e.: $\max _{p_{1}}\left(p_{1}-5\right)\left(20-\frac{4}{3} p_{1}+\frac{2}{3} \frac{40+p_{1}}{4}\right)=\left(p_{1}-5\right)\left(\frac{160-7 p_{1}}{6}\right)$

FOC: $p_{1}=\frac{195}{14}=13,928$
$p_{2}=\frac{40+13,928}{4}=13,482$
II. 4 Consider the following 2-player static game, where Player 1 has strategies $a$ and $b$ and Player 2 has strategies $x$ and $y$, defined by payoff matrix:

|  | $x$ | $y$ |
| :---: | :---: | :---: |
| $a$ | 4,3 | 0,4 |
| $b$ | 5,0 | 1,1 |

Consider that decisions are made once a month and that the game lasts for two months.
(a) How many information sets owns each player? How many strategies has each player? How many subgames are there? (2 points)
(b) Which is the subgame perfect Nash equilibrium (SPNE) of the game? Which are the payoffs of the players in the ENPS? (4 points)

Consider now that the game lasts indefinitely and that both players have the same discount rate given by $\delta$.
(c) Which is the smallest discount rate $\delta$ that can sustain $(a, x)$ in a subgame perfect equilibrium using "trigger" strategies? (14 points)
(a) Each player has 5 information sets ( 1 in the first stage and 4 in the 2 nd ). Having 2 possible accions in each information set the total number of strategies for each player $2^{5}=32$. The number of proper subgames is 4 (if we include the whole game, 5).
(b) In the stage game, there is only one NE, $(b, y)$, with payoffs $(1,1)$. The only SPNE when the game is repeated finitely many times is top play the NE of the game in each stage. I.e., the SPNE is (bbbbb, yyyyy) and playoffs would be $(2,2)$.
(c) d) If we want that J1 plays $a$ and does not deviate to $b$, we need:
$4+4 \delta+4 \delta^{2}+4 \delta^{3}+\cdots \geq 5+\delta+\delta^{2}+\delta^{3}+\cdots$
From that inequality we get $\delta \geq \frac{1}{4}$.
If we want that J 2 plays $x$ and does not deviate to $y$, we need:
$3+3 \delta+3 \delta^{2}+3 \delta^{3}+\cdots \geq 4+\delta+\delta^{2}+\delta^{3}+\cdots$
From there: $\delta \geq \frac{1}{3}$.
Thus the smallest discount rate that satisfies both restrictions is $\delta=\frac{1}{3}$.
II.5 Firm Speak More Telecom wants to renew its network of antennae for cell phone communications, and is proposing an auction between two providers, Solutions Without Cables, SWC, and Telecommunications Without Wires, TWW. The auction has the following rules:

- Each firm can bid 1,200 or 900 monetary units per antenna.
- The firm with the lowest bid gets the contract and, in case of a tie, the contract is assigned at random with probabilities $1 / 2,1 / 2$.
- The winner is paid the quantity in his bid per antenna.

Both firms know that TWW has a production cost of 700 per unit. It is also known that SWC may have a unit cost of 600 or 1,000 with equal probabilities. SWC knows his costs, but TWA does not know SWC's costs.
(a) Describe this situation as a Bayesian game. I.e., show the set of players and of types of players. Show also each type's beliefs about types of other players, the set of strategies of types and the utilities in the usual matrix form. (10 points)
(b) Find the Bayesian Nash equilibria of the game. (10 points)
(a) Players $=\{\mathrm{SWC}, \mathrm{TWW}\}$

Types of SWC $=\left\{\mathrm{SWC}_{600}, \mathrm{SSC}_{1000}\right\}$, types of TWW $=\{\mathrm{TWW}\}$
Probabilities: $p\left(\mathrm{SWC}_{600} / \mathrm{TWW}\right)=\frac{1}{2}, p\left(\mathrm{SWC}_{1000} / \mathrm{TWW}\right)=\frac{1}{2} ; p\left(\mathrm{TWW} / \mathrm{SWC}_{600}\right)=1$; $p\left(\mathrm{TWW} / \mathrm{SWC}_{1000}\right)=1$.
Strategies for SWC $=\{(1200,1200),(1200,900),(900,1200),(900,900)\}$.
Strategies for TWW $=\{1200,900\}$.
Payoffs:

|  | $\left(\mathrm{SWC}_{600}\right)$ | TWW |  | $\left(\mathrm{SWC}_{1000}\right)$ | TWW |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | 1200 | 900 |  | 1200 | 900 |
| SWC | 1200 | 300, 250 | 0,200 |  | 100, 250 | 0,200 |
|  | 900 | 300, 0 | 150, 100 |  | -100, 0 | -50, 100 |

(b) $\mathrm{SWC}_{1000}$ bids 1200 (it dominates 600).
$\mathrm{BNE}=\{((1200,1200),(1200)),(900,1200),(900))\}$.

