## Game Theory <br> January 2017 exam

Name:

## Group:

## You have two and a half hours to complete the exam.

## I. Short questions (5 points each)

I. 1 Give an example of a static game where weakly elimination of a strategy causes to eliminate one of the Nash equilibria.

| 1,1 | 0,0 |
| :--- | :--- |
| 0,0 | 0,0 |

I. 2 Two players participate in a first-price, closed-envelope auction. If they both bid the same quantity, they get the item with probability $1 / 2$. Players' valuations are 10 and 11 , respectively. Only integer numbers are permitted in the bids. Find the three Nash equilibria in pure strategies.

$$
\mathrm{EN}=\{(9,9),(9,10),(10,10)\}
$$

I. 3 In the following game describe the strategy sets for the players (the first node corresponds to a nature or chance move). How many strategies has each player?


Player 1: $S_{1}=\{(x, x),(x, y),(y, x),(y, y)\}$. Four.
Player 2: $S_{2}=\{l, r\}$. Two.
I. 4 Consider a Cournot duopoly with asymmetric information where Firm 2 does not know whether the costs of Firm 1 are high or low. If Firm 1 had high costs, would it prefer the situation with symmetric information? What if it had low costs?

High costs Firm 1 prefers costs not to be known (asymmetric information.)
Low costs Firm 1 prefers costs to be known (symmetric information.)

## II. Problems (20 points each)

II. 1 Among the possible externalities that affect economic decisions, and that pull them afar from efficiency, there is the case when utility depends not only on absolute consumption, but also on relative consumption. Let us see one such a case. Say there are two consumers, 1 and 2, with preferences given by $u_{i}=x_{i}\left(16-l_{i}\right)$, where $x_{i}$ is the quantity of $x$ consumed during a day, and $l_{i}$ is the amount of time devoted to work $(i=1,2)$, measured in hours a day $\left(16-l_{i}\right.$ is leisure time after discounting the sleeping time). To make the example simple, say that the salary per hour is $€ 10$, and that the price per unit of the consumption good is also $€ 10$. In addition, there is no possibility of saving.
(a) Solve the consumers' problems. Find the number of hours per day that they will dedicate to work and their utilities. (5 points.)

Let us add a new term to the utilities that takes into account the preferences for relative consumption: $u_{i}=x_{i}\left(16-l_{i}\right)+4\left(x_{i}-x_{j}\right)$. In this function, we observe that if own consumption is greater than the other consumer's, $x_{i}>x_{j}$, then the utility increases, while if $x_{i}<x_{j}$ it decreases. With these new utility functions answer the following questions.
(b) Repeat the questions in (a). (5 points.)
(c) Construct the following simultaneous game: each consumer must decide between working the hours found in (a) and the hours found in (b). Find the Nash equilibrium. Is it optimal? (10 points.)
(a) Consumer $i$ solves

$$
\begin{aligned}
& \max _{x_{i}, l_{i}} u_{i}=x_{i}\left(16-l_{i}\right) \\
& \text { s.a } 10 x_{i}=10 l_{i}
\end{aligned}
$$

Or

$$
\max _{l_{i}} u_{i}=l_{i}\left(16-l_{i}\right)
$$

whose solution is $l_{i}=8$. From there: $x_{i}=8$ y $u_{i}=64$.
(b) The new problem is

$$
\begin{aligned}
& \max _{x_{i}, l_{i}} u_{i}=x_{i}\left(16-l_{i}\right)+4\left(x_{i}-x_{j}\right) \\
& \text { s.a } 10 x_{i}=10 l_{i}
\end{aligned}
$$

Or

$$
\max _{l_{i}} u_{i}=l_{i}\left(16-l_{i}\right)+4\left(l_{i}-x_{j}\right)
$$

whose solution is $l_{i}=10$. From there $x_{i}=10$ y $u_{i}=60$.
(c) The game is

Firm 2

|  |  | $l_{2}=8$ | $l_{2}=10$ |
| :---: | :---: | :---: | :---: |
|  |  |  |  |
| Firm 1 | $l_{1}=8$ | 64,64 | 56,68 |
|  | $l_{1}=10$ | 68,56 | 60,60 |
|  |  |  |  |

The equilibrium $\left(l_{1}=10, l_{2}=10\right)$ is note optimal: the pair of strategies $\left(l_{1}=8, l_{2}=8\right)$ is better for both.
II. 2 Two firms compete a la Bertrand with differentiated products. Demands are given by:

$$
\begin{aligned}
& q_{1}=10 a-2 p_{1}+p_{2} \\
& q_{2}=10 a-2 p_{2}+p_{1}
\end{aligned}
$$

where $a \geq 0$ is a parameter that takes real values. To simplify, assume that production costs are zero for both firms. In the first stage, Firm 1 can start an advertising campaign that will attract consumers to the market, determining the value of $a$. In particular, Firm 1 has to pay $c(a)=a^{3}$ to get that the parameter takes the value $a$. In the second stage, knowing the value of $a$, firms compete a la Bertrand, choosing prices simultaneously.
(a) Describe the situation as a dynamic game. How many subgames are there in this game? (4 points.)
(b) Find the Nash equilibria of the subgames that start in the second stage. (7 points.)
(c) Find the subgame perfect Nash equilibria of the whole game. (9 points.)
(a)


There is a subgame for every value of $a$ chosen by Firm 1, which means infinitely many of them.
(b) Firm 1 solves $\max _{p_{1}} \Pi_{1}=\left(10 a-2 p_{1}+p_{2}\right) p_{1}-a^{3}$, whose first order condition is $\frac{\partial \Pi_{1}}{\partial p_{1}}=10 a-4 p_{1}+p_{2}=0$ and from there we obtain the best reply function $p_{1}=\frac{10 a+p_{2}}{4}$. (The second order condition, $\frac{\partial^{2} \Pi_{1}}{\partial\left(p_{1}\right)^{2}}=-4<0$, confirms it is a maximum). Firm 2 solves $\max _{p_{2}} \Pi_{2}=$ $\left(10 a-2 p_{2}+p_{1}\right) p_{2}$, whose f.o.c. gives $p_{2}=\frac{10 a+p_{1}}{4}$. (The s.o.c. is analogous to that of Firm 1). With the best reply conditions we get NE: $\left(p_{1}=\frac{10 a}{3}, p_{2}=\frac{10 a}{3}\right)$.
(c) Firm 1 solves $\max _{a} \Pi_{1}=\left(10 a-2 \frac{10 a}{3}+\frac{10 a}{3}\right) \frac{10 a}{3}-a^{3}$, whose f.o.c. is $\frac{\partial \Pi_{1}}{\partial a}=\frac{400 a}{9}-3 a^{2}=$ 0 , that gives two critical points: $a=0$ and $a=\frac{400}{27}$. The maximum profit is obtained at $a=\frac{400}{27}$. The s.o.c. for this value is $\frac{\partial^{2} \Pi_{1}}{\partial a^{2}}=\frac{400}{9}-6 \frac{400}{27}<0$, that confirms it is a local maximum. The only corner solution, $a=0$, has a smaller value, and the profits function is decreasing to the right of $a=\frac{400}{27}$ (the first derivative is negative), so it is a global maximum.
II. 3 It is Saturday evening and Jim and Buzz, each in a car, drive towards a cliff. The first one stopping will be a coward (chicken), and the other one will be the winner. If they both keep driving, they will go down the cliff, but if they both stop they will be less coward relatively speaking. The game has the following strategies and payoffs ("keep going" means stop only after the other has already stopped).

(a) Find the three Nash equilibria of the game and the utilities of the players. (4 points.)
(b) Consider the situation in which Jim and Buzz play the game this Saturday and the next one. Is it possible that they both stop first this Saturday in a subgame perfect Nash equilibrium? Consider that there is no time discount in this version of the game. Hint: study separately each one of the four subgames of the second stage. I.e.: what must the equilibrium strategy propose to each player in each one of the four possibilities so that no one wants to deviate from (Stop 1st, Stop 1st) in the first stage? (8 points.)
(c) Consider that Jim and Buzz play the game each Saturday indefinitely. Is it possible to sustain the payoff $(3,3)$ every Saturday in a subgame perfect Nash equilibrium? In this version, both players use a time discount rate of $\delta$. (8 points.)
(a) $\mathrm{EN}=\left\{(\mathrm{Go}\right.$, Stop $),($ Stop, Go $),\left(\frac{1}{2}[\mathrm{Go}]+\frac{1}{2}[\right.$ Stop $], \frac{1}{2}[\mathrm{Go}]+\frac{1}{2}[$ Stop $\left.\left.]\right)\right\}$. The respective utilities are $(4,1),(1,4)$ and $(2,2)$.
(b) First Saturday: play (Stop, Stop). Second Saturday:
(i) play $\left(\frac{1}{2}[\mathrm{Go}]+\frac{1}{2}[\right.$ Stop $], \frac{1}{2}[\mathrm{Go}]+\frac{1}{2}[$ Stop $\left.]\right)$ if they played (Stop, Stop) or (Go, Go) the first Saturday,
(ii) play (Go, Stop) if they played (Stop, Go),
(iii) play (Stop, Go) if they played (Go, Stop).

The strategy is a NE in each proper subgame, let us check that it is also a NE in the whole game. If they both follow the strategy, each gets: $u_{i}=3+2=5$. If one deviates, he gets $4+1=5$, so no gain is obtained and the strategy is a NE.
(c) The simplest (but not the only one) is: First Saturday, play (Stop, Stop). Next Saturdays:
(i) play (Stop, Stop) if they played (Stop, Stop) all previous Saturdays,
(ii) play $\left(\frac{1}{2}[\mathrm{Go}]+\frac{1}{2}\right.$ [Stop], $\frac{1}{2}[\mathrm{Go}]+\frac{1}{2}[$ Stop $\left.]\right)$ if at any Saturday they did not play (Stop, Stop).

The strategy is a Nash equilibrium: if followed, each player gets $u_{i}=3+3 \delta+3 \delta^{2}+\cdots=$ $3 \frac{1}{1-\delta}$. The best deviation is in one period (explain), that gives $u_{i}=4+2 \delta+2 \delta^{2}+\cdots=4+$ $2 \frac{\delta}{1-\delta}$ to the deviator. Fort he deviation not to be profitable it must hold that $3 \frac{1}{1-\delta} \geq 4+2 \frac{\delta}{1-\delta}$, or: $\delta \geq \frac{1}{2}$.
The strategy is a NE in subgames in case (i): the game and the strategy in these subgames are the same as in the whole game, so it is a NE.
The strategy is a NE in subgames in case (ii): the strategy defines a NE in each repetition, which constitutes a NE of the repeated game.
II. 4 The countries of Anchuria and Borduria are involved in a dispute over a piece of oil-rich territory. The value of the oil is 100 billion. Each country has two options: escalate the conflict, or negotiate. Decisions are made simultaneously. If both countries decide to negotiate, the oil will be split equally (thus, each country will receive 50 billion). If one escalates and the other negotiates, the country that escalates will receive all the oil, and the other country will receive nothing. If both decide to escalate, a war starts, and the winner receives all the oil, while the loser receives nothing. Each country has an equal chance of winning the war. The war, however, is costly. Both countries know that the war will cost Anchuria 30 billion. The cost to Borduria depends on its military strength, which only Borduria knows. If Borduria is strong, the war will cost it 10 billion. If Borduria is weak, the war will cost it 50 billion. Borduria is equally likely to be strong and weak.
(a) Represent the situation as a Bayesian game. That is, show the set of players and the players' types, the beliefs about the types of other players, and the set of actions for each player. Show also the players' utilities in the usual matrix form. (4 points.)
(b) Find all pure-strategy Bayesian Nash equilibria of the game. (8 points.)
(c) Suppose now that if Borduria is strong it will win the war with probability one, but if it is weak, it will lose also with probability one. Everything else remains the same. Find all the pure-strategy Bayesian Nash equilibria. Find also the types' utilities in the equilibria you found. (8 points.)
(a) $N=\{A, B\}$, where $A=$ Anchuria and $B=$ Borduria.
$T_{A}=\{A\} ; T_{B}=\left\{B_{s}, B_{w}\right\}$, where $s$ stands for strong and $w$ stands for weak.
$p\left(A / B_{S}\right)=1 ; p\left(A / B_{w}\right)=1 ; p\left(B_{S} / A\right)=\frac{1}{2}, p\left(B_{w} / A\right)=\frac{1}{2}$.
$A$ 's actions: $A_{A}=\{\mathrm{N}, \mathrm{E}\}$, where N means to negotiate and E means to escalate. Actions of $B_{s}: A_{B_{s}}=\{\mathrm{N}, \mathrm{E}\}$, actions of $B_{w}: A_{B_{w}}=\{\mathrm{N}, \mathrm{E}\}$. Strategies of $A: S_{A}=\{\mathrm{N}, \mathrm{E}\}$, strategies of $B$ :
$S_{B}=\{(\mathrm{N}, \mathrm{N}),(\mathrm{N}, \mathrm{E}),(\mathrm{E}, \mathrm{N}),(\mathrm{E}, \mathrm{E})\}$, where the first component in each strategy indicates the action by $B_{s}$ and the second, the action by $B_{w}$.
Utilities:
$B_{s}$ with $p=\frac{1}{2} \quad B_{s} \quad B_{w}$ with $p=\frac{1}{2} \quad B_{w}$
(b) For $A, \mathrm{E}$ is a dominant strategy. Para $B_{s}, \mathrm{E}$ is a dominant action. $E N B=\{\mathrm{E},(\mathrm{E}, p[\mathrm{~N}]+$ $(1-p)[E])$ for all $p \in[0,1]\}$.
(c) The new game is:
$B_{S}$ with $p=\frac{1}{2} \quad B_{S} \quad B_{w}$ with $p=\frac{1}{2} \quad B_{w}$

$$
\begin{aligned}
&
\end{aligned}
$$

$B N E=\{\mathrm{E},(\mathrm{E}, \mathrm{N})\} . B$ is clearly using its best reply against the choice of E by A . Let us see the case of $A$ : if $B$ plays $(E, N), u_{A}(N,(E, N))=\frac{1}{2} 0+\frac{1}{2} 50=25$, while $u_{A}(E,(E, N))=\frac{1}{2}(-30)+$ $\frac{1}{2} 100=35$, which is bigger.

If $A$ chooses $N$, the best reply by $B$ is $(N, N)$, but $A$ 's the best reply against $(E, E)$ is not $N$, which gives 0 ; but $E$, which gives $\frac{1}{2}(-30)+\frac{1}{2} 70=20$. So there is no other BNE in pure strategies.

