Game Theory
January 2018 exam

Name: **Group:**

You have two and a half hours to complete the exam. No calculators.

I. Short questions (5 points each)

I.1 Can there be a static game in which every strategy is rationalizable? If yes, provide an example. If no, explain why.

Yes:

1, -1	-1, 1
-1, 1	1. -1

I.2 Give an example of a dynamic game where the player who moves first has an advantage, and an example where it is the player moving last the one that has an advantage.

-Stackelberg duopoly
-Bertrand duopoly with differentiated goods

I.3 In a bargaining game with alternating offers, give a list of the features that improve a player's equilibrium payoffs, and explain briefly why.

-Playing last with few rounds.
-Not being risk averse.
-Being patient.

I.4 Find the Bayesian equilibria in pure strategies of the following Bayesian game, where players are in game (a) or (b) with probabilities ½, ½. Player 1 knows in which game they are, but Player 2 only knows it is one of them with the abovementioned probabilities.

(a)

		Player 2: A	Player 2: B
Player 1	C	1, 1	0, 0
	D	0, 0	-1, -1

(b)

		Player 2: A	Player 2: B
Player 1	C	-1, -1	0, 0
	D	0, 0	1, 1

Player 1 plays C in (a) and D in (b) as they are both dominant strategies. Player 2's best response: if she plays A she gets: $\frac{1}{2}1 + \frac{1}{2}0 = 0.5$, if she plays B: $\frac{1}{2}0 + \frac{1}{2}1 = 0.5$. She is indifferent. BNE in pure strategies ={((C,D),A), ((C,D), B)}.

II. Problems (20 points each)

II.1 Two countries, Aldorria and Borginia, are involved in a dispute over a border province that is rich in natural resources. The value of the province is 100. The two countries simultaneously choose whether to escalate the dispute by sending troops to the border, or to back down by seeking a diplomatic solution. If both countries back down, the dispute is settled through diplomacy, and each country gets half of the province. If one country escalates and the other backs down, the one that escalates captures the entire province while the other receives nothing. If both countries escalate, a war starts between them. The war will result in each country capturing half of the border province. In addition, the war will cost each country $X > 0$.

(a) Represent this situation as a static game. (4 points)
(b) Find the Nash equilibria in pure strategies for the different values of *X*. (8 points)

The UN wants to encourage the two countries to resolve the dispute through diplomacy. To do so, it imposes sanctions as a punishment for escalating the dispute. As a result, the country that escalates will suffer an additional loss that equals *Y*.

(c) Represent the new payoff matrix. How large should *Y* be to ensure that the only equilibrium is for both countries to back down, regardless the value of *X*? (8 points)

(a) $N = \{\text{Aldorria , Borginia}\}$, $S_A = S_B = \{\text{Escalate, Back down}\}$, payoffs are given by the matrix:

		Borginia	
		Escalate	Back down
Aldorria	Escalate	$50 - X, 50 - X$	100, 0
	Back down	0, 100	50, 50

(b)

If $X < 50$, (Escalate, Escalate) is the only Nash equilibrium.

If $X \geq 50$, $NE = \{(\text{Escalate, Back down});\ (\text{Back down, Escalate});\ (p[\text{Escalate}] + (1 - p)[\text{Back down}]\ , q[\text{Escalate}] + (1 - q)[\text{Back down}])$ where $p = q = 50/X\}$.

The last result also includes a special case in which $X = 50$, and the NE are (Escalate, Back down); (Back down, Escalate); and (Escalate, Escalate).

(c)

		Borginia	
		Escalate	Back down
Aldorria	Escalate	$50 - X - Y, 50 - X - Y$	$100 - Y, 0$
	Back down	$0, 100 - Y$	50, 50

(Back down, Back down) is a unique equilibrium if and only if $Y > 50$.

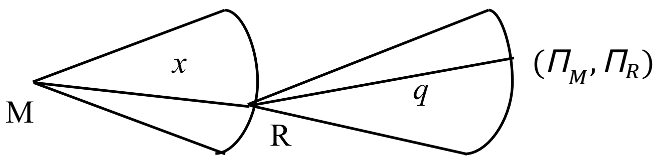
II.2 A manufacturer of automobile tires produces at a cost of €10 per tire. It sells units to a retailer who in turn sells the tires to consumers, whose inverse demand function is $p = 200 - \frac{q}{100}$. The retailer has no production costs other than whatever it must pay to the manufacturer for the tires. Suppose that the manufacturer and retailer interact as follows. First, the manufacturer sets a price x that the retailer must pay for each tire. Then, the retailer decides how many tires q to purchase from the manufacturer and sell to consumers at a price set by the demand function.

(a) Write down the profit functions for both the manufacturer and the retailer. (4 points)
(b) Calculate the subgame perfect Nash equilibrium of this game. (8 points)

Suppose that the manufacturer sells its tires directly to the consumers, bypassing the retailer.

(c) Calculate the manufacturer's profit-maximizing number of tires to sell. (4 points)
(d) Compare the profits for the manufacturer in the two situations. Compare also the prices for the consumers. (4 points)

(a)



The manufacturer's payoff (profit) is: $q(x-10)$. The retailer's profit is: $\left(200 - \frac{q}{100}\right)q - xq$.

(b) The first-order condition for optimization implies that the retailer chooses quantity

$$q^*(x) = 10{,}000 - 50x.$$

Thus, the manufacturer can anticipate selling exactly $q^*(x)$ units if it prices at x, which means that the manufacturer's payoff, as a function of x,

$$q^*(x)(x-10) = (10{,}000 - 50x)(x-10) = 10{,}500x - 50x^2 - 100{,}000.$$

Maximizing the expression gives $10{,}500 - 100x = 0$, or $x^* = 105$. The subgame perfect equilibrium is: ($x^* = 105$, $q^*(x) = 10{,}000 - 50x$).

The equilibrium quantity is $q = q^*(105) = 4750$. Thus, the equilibrium path is ($x^* = 105$, $q^*(105) = 4750$). This implies $p = 152.50$.

(c) In this case, the manufacturer's profit is:

$$\left(200 - \frac{q}{100}\right)q - 10q.$$

Taking the derivative of this expression and setting it equal to zero yields the manufacturer's optimal quantity, which is $q^* = 9{,}500$. This implies $p^* = 105$.

(d) Retailer's profits in (b) are $4750 \times (152.5 - 105) = 4750 \times 47.5 = 225{,}625$.
Manufacturer's profits in (b) are $4750 \times (105 - 10) = 451{,}200$.
Manufacturer's profits in (c) are $9500 \times (105 - 10) = 902{,}500$.
The manufacturer makes more profits in (c), without the retailer. Consumers see a lower price 105 rather than 152.5, and buy a higher quantity, 9,500 rather than 4,750, also without the retailer.

II.3 Consider the infinite repetition of the following game, where players calculate payoffs using the same discount rate, δ.

		She	
		Football	Theater
He	Football	1, 6	2, 2
	Theater	2, 2	6, 1

Players wish to obtain the following equilibrium path: in odd periods play (Football, Football), and in even periods play (Theater, Theater). The game starts in period 1.

(a) What is the utility for each player if they follow the desired path? Hint: calculate the utility grouping the odd periods on one side, and the even periods on the other. (5 points)
(b) Describe the trigger strategy that is a candidate to sustain the path above in a subgame perfect Nash equilibrium. (5 points)
(c) Which is the minimum value of δ so that He does not want to deviate from the trigger strategy in period 1? And for She in period 2? (5 points)
(d) Show that the trigger strategy is indeed a subgame perfect Nash equilibrium for values of δ equal or higher that the maximum of the ones found in (c). (5 points)

(a) $u_{He} = 1 + 6\delta + \delta^2 + 6\delta^3 + \delta^4 + 6\delta^5 + \cdots =$
$1 + \delta^2 + \delta^4 + \cdots + 6\delta + 6\delta^3 + 6\delta^5 + \cdots =$
$\frac{1}{1-\delta^2} + \frac{6\delta}{1-\delta^2} = \frac{1+6\delta}{1-\delta^2}$.

$u_{She} = 6 + \delta + 6\delta^2 + \delta^3 + 6\delta^4 + \delta^5 + \cdots =$
$6 + 6\delta^2 + 6\delta^4 + \cdots + \delta + \delta^3 + \delta^5 + \cdots =$
$\frac{6}{1-\delta^2} + \frac{\delta}{1-\delta^2} = \frac{6+\delta}{1-\delta^2}$.

(b) In $t = 1$ play (Football, Football).
In $t > 1$ play:
(i) (Football, Football) if $t = odd$, and for all past periods players played (Football, Football) in odd periods and (Theater, Theater) in even periods.
(ii) (Theater, Theater) if $t = even$, and for all past periods players played (Football, Football) in odd periods and (Theater, Theater) in even periods.
(iii) (Theater, Football) if at some point in the past players did not play (Football, Football) in odd periods and (Theater, Theater) in even periods.

(c) If He deviates in period 1: $u_{He} = 2 + 2\delta + 2\delta^2 + 2\delta^3 + \cdots = \frac{2}{1-\delta}$. The deviation is no good if $\frac{1+6\delta}{1-\delta^2} \geq \frac{2}{1-\delta}$, which means $\frac{1+6\delta}{1-\delta^2} \geq \frac{2(1+\delta)}{1-\delta^2}$, or $1 + 6\delta \geq 2 + 2\delta$), which implies $\delta \geq \frac{1}{4}$.

If She deviates (in period 2) She's utility will be: $u_{She} = 6 + 2\delta + 2\delta^2 + 2\delta^3 + \cdots = 6 + \frac{2\delta}{1-\delta}$. The deviation is no good if $\frac{6+\delta}{1-\delta^2} \geq \frac{2}{1-\delta}$, which implies $\delta \geq \frac{1}{4}$. Alternatively, you can argue that She's utility and deviation evaluated at period 2 are exactly the same as He's evaluated at period 1.

(d) (i) The trigger strategy is a NE in the whole game.

The best deviations for He are one-period deviations in an odd period because in an odd period, He loses, and deviations for more than one day are worthless, as after the first deviation, the trigger strategy requires players play the Nash equilibrium unconditionally for ever after. The same goes for She in an even period. In any odd (even) period, He's (She's) deviations profitability are computed as in (c), which means that the strategy is a NE in the game for $\delta \geq \frac{1}{4}$.

(ii) The trigger strategy is a NE in periods after players alternated between (Football, Football) in odd periods and (Theater, Theater) in even periods.

The game in odd periods is exactly as the original game, and the trigger strategy says the same. So the trigger strategy is a NE in those subgames. The game in even periods is exactly as the game starting in $t = 2$. We already saw that in this period She's best deviation is not worthwhile if $\delta \geq \frac{1}{4}$. So the trigger strategy is a NE in those subgames also.

(iii) The trigger strategy is a NE in periods after players did not alternate between (Football, Football) in odd periods and (Theater, Theater) in even periods.

In these periods, the strategy requires to play the only NE in all periods, unconditionally, and that is a NE in all those subgames.

II.4 Consider a Cournot duopoly with two firms operating in a market where the inverse demand function is $P = A - Q$, where Q is the total output in the market. Suppose that there is uncertainty about the value of A, which with probability $p = \frac{1}{2}$ can be high, $A = 24$, and with probability $p = \frac{1}{2}$ can be low, $A = 12$. The production costs are 0 for both firms. Firm 2 is informed about the value of A, but Firm 1 only knows the above probabilities. Both firms decide independently and simultaneously their respective output, q_1 and q_2.

(a) Describe the above situation as a Bayesian game. (5 points)
(b) Compute the Bayesian equilibrium. (8 points)
(c) Assume now that there is no uncertainty and that both firms know that the market demand is medium, $A = 18$. Compute the Cournot equilibrium. (4 points)
(d) Compare the profits in cases (b) and (c) for both firms. (3 points)

(a)
$N = \{\text{Firm 1, Firm 2}\}$;
$T_1 = \{t_1\}, T_2 = \{t_2^H, t_2^L\}$;
$(p(t_2^H|t_1) = \frac{1}{2}, p(t_2^L|t_1) = \frac{1}{2}), (p(t_1|t_2^H) = 1), p(t_1|t_2^L) = 1.$
$A_1 = \{q_1 \in [0, \infty)\}, A_{t_2^H} = \{q_H \in [0, \infty)\}, A_{t_2^L} = \{q_L \in [0, \infty)\}.$
$S_1 = A_1, S_2 = A_{t_2^H} \times A_{t_2^L}.$
$u_1(q_1, q_H, q_L) = \frac{1}{2}(24 - q_1 - q_H)q_1 + \frac{1}{2}(12 - q_1 - q_L)q_1.$
$u_H(q_1, q_H, q_L) = (24 - q_1 - q_H)q_H,$
$u_L(q_1, q_H, q_L) = (12 - q_1 - q_H)q_L.$

(b) The best reply functions are

$q_1(q_H, q_L) = \frac{1}{4}(36 - q_H - q_L),$
$q_H(q_1) = \frac{24 - q_1}{2},$
$q_L(q_1) = \frac{12 - q_1}{2}.$

The BNE is $(q_1 = 6, q_H = 9, q_L = 3)$.

(c) Best reply functions are $q_1 = \frac{18 - q_2}{2}, q_2 = \frac{18 - q_1}{2}$, the solution gives the NE: $q_1 = q_2 = 6$.

(d) Profits in (b) are $\Pi_1 = 36, \Pi_H = 81, \Pi_L = 9$.
Profits in (c) are $\Pi_1 = \Pi_2 = 36$.

Firm 1 is indifferent between situations (b) and (c). In (b) Firm 2 gets expected profits $E(\Pi_2) = 45 > 36$. Thus, it will be better off in that case.