Game Theory
Exam January 2015

Name **Group:**

You have two and a half hours to complete the exam.

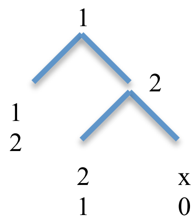
I. Short questions (5 points each)

I.1 Provide an example of a static game in which the elimination of weakly dominated strategies also eliminates a Nash equilibrium.

	L	R
U	1,1	0,0
D	0,0	0,0

D is weakly dominated by U, R is weakly dominated by L, (D,R) is a Nash equilibrium.

I.2 Consider the following game



For which values of x has the game only one Nash equilibrium?
For which values of x has the game two Nash equilibria, but only one of them is a subgame perfect Nash equilibrium?

	L	R
U	1,2	1,2
D	2,1	x,0

For $x > 1$ the only NE is (D,L)

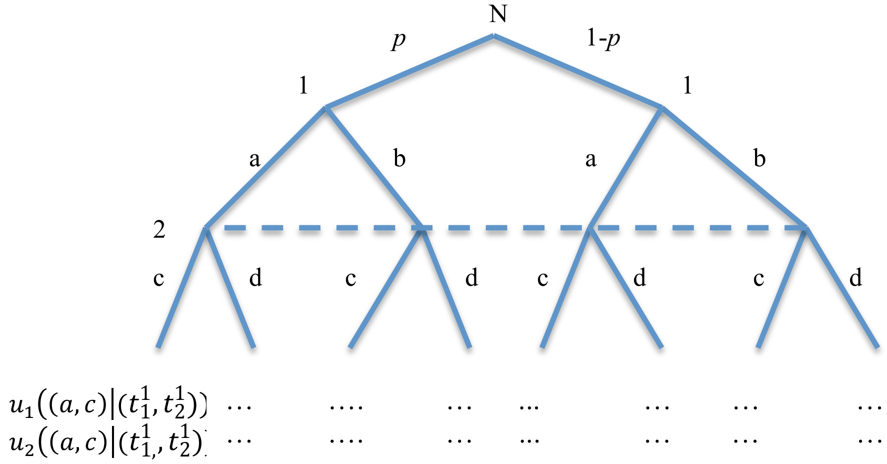
For $x < 1$ both (D,L) and (U,R) are NE, but only (D,L) is a SPNE

I.3 Describe the trigger strategy to support the cooperative outcome in a prisoners' dilemma repeated infinitely many times.

s_i = Cooperate at $t = 1$, cooperate at $t > 1$ if (cooperate, cooperate) was played at all $t' < t$, otherwise do not cooperate. (Also good: cooperate at $t > 1$ if (cooperate, cooperate) was played at all $t - 1$.)

I.4 Define completely a Bayesian game of two players with two types of Player 1 and one type of Player 2. Draw its extensive form representation.

$N = \{1,2\}$, $T_1 = \{t_1^1, t_1^2\}$, $T_2 = \{t_2^1\}$, $(p(t_2^1|t_1^1) = p$, $p(t_2^2|t_1^1) = 1 - p)$, $p(t_1^1|t_2^1) = 1$, $p(t_1^1|t_2^2) = 1$, $A_{t_1^1} = \{a, b\}$, $A_{t_2^1} = A_{t_2^2} = \{c, d\}$. Utilities as shown:



II. Problems (20 point each)

II.1 Two theaters are located on the same street. They can advertise and by doing so attract customers to the area so that each of the two theaters' advertising is beneficial to both. Let x_1 be the advertising level of Theater 1, and x_2 be the advertising level of Theater 2. The profit functions for theaters 1 and 2 are, respectively:

$$\Pi_1(x_1, x_2) = (30 + x_2)x_1 - 2(x_1)^2$$
$$\Pi_2(x_1, x_2) = (30 + x_1)x_2 - 2(x_2)^2.$$

(a) What are the Nash equilibrium advertising levels of the two theaters? (8 points)
(b) What are their equilibrium profits? (2 points)
(c) Are there advertisement levels that give both theaters higher profits than the ones in the equilibrium? Which levels give the highest profits? (5 points)
(d) Why advertisement levels in (c) are not a Nash equilibrium? (5 points)

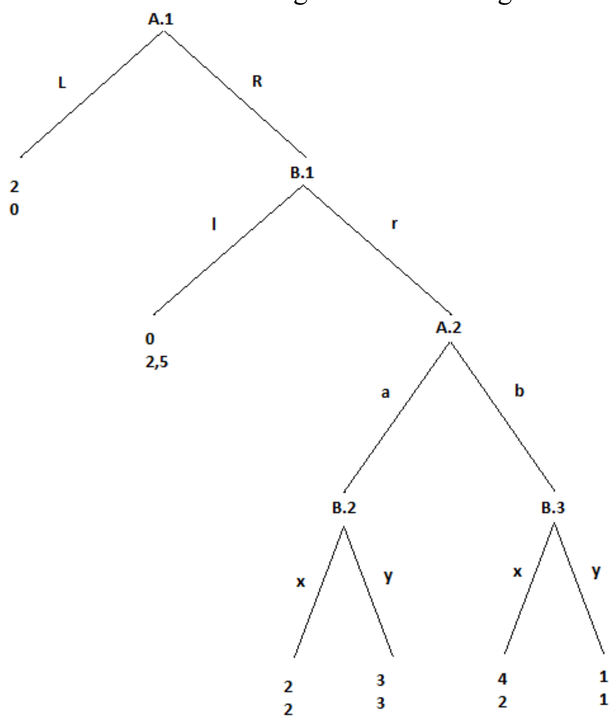
(a) $Max_{x_1 \geq 0}(30 + x_2)x_1 - 2(x_1)^2$
FOC: $30 + x_2 - 4x_1 = 0$
Same for 2: $30 + x_1 - 4x_2 = 0$
Solving the two equations: $x_1 = x_2 = 10$. NE = $\{(10,10)\}$.

(b) $\Pi_i(10,10) = (30 + 10)10 - 2(10)^2 = 200$ for *i* = 1,2.

(c) Highest profits: $Max_{x_1 \geq 0, x_2 \geq 0}(30 + x_2)x_1 - 2(x_1)^2 + (30 + x_1)x_2 - 2(x_2)^2$.
FOC: $30 + x_2 - 4x_1 + x_2 = 0$,
$30 + x_1 - 4x_2 + x_1 = 0$.
Solving: $x_1 = x_2 = 1$
Profits are: $\Pi_i(15,15) = (30 + 15)15 - 2(15)^2 = 225$

(d) Not NE because if $x_2 = 15$, the best reply for 1 is $x_1 = \frac{30+x_2}{4} = 11.25$,
with profits $\Pi_1(11.25,15) = (30 + 15)11.25 - 2(11.25)^2 = 253.125$.

II.2 Consider the following extensive form game.



(a) Write down both players' strategy sets. (3 points)

(b) Find the subgame perfect Nash equilibria of the game. (7 points)

For the rest of this exercise, suppose that when Player B has to decide between x and y he does not know whether Player A has chosen a or b at A.2.

(c) Find the Nash equilibria of the subgame that starts at A.2. (5 points)
(d) Find the subgame perfect Nash equilibria of the entire game. (5 points)

(a) Strategy set of player A: {La,Lb,Ra,Rb },
Strategy set of player B: {lxx,lxy,lyx,lyy, rxx,rxy,ryx,ryy}.

(b) SPNE = {(Lb, lyx)}.

(c) NE={(a,y),(b,x),(1/2[a] + 1/2[b], 1/2[x]+ 1/2[y])}.

(d) First we find the payoffs of the two players for each of the NE we found in part (c): (3,3), (4,2) and (2,5 ,2). At B.1 player B will choose r only if the NE that follows is the one that gives him at least 3, which is (a,y). He will choose l in the other two cases. Therefore the SPNE of a game starting at B.1 are: (a,ry), (1/2[a] + 1/2[b], (l, 1/2[x]+ 1/2[y])) and (b,lx). The payoffs of player A for each of these are: 3, 0 and 0. This means that the SPNE = {(Ra,ry), ((L, 1/2[a] + 1/2[b]), (l, 1/2[x]+ 1/2[y])), (Lb,lx)}.

II.3 Pitita and Pedro Luis just got married in a country with no chance of divorce. Each period, they have to decide whether to help each other or not. Help, denoted by q, is measured in the interval [0,1]. A help amount of q represents a cost of $2q$ to the helper and a profit of $3q$ to the one being helped.

(a) Compute the Nash equilibrium if the game lasts only one period. (5 points)
(b) Compute the subgame perfect Nash equilibria if the game lasts T periods. (5 points)
(c) Suppose now that both Pitita and Pedro Luis hope to live forever. Compute the discount factor so that the socially optimal level of help is supported in a subgame perfect Nash equilibrium. (10 points)

(a) No one helps, it is a strictly dominant strategy: NE = {(0,0)}.

(b) The SPE is the unconditional repetition of the only NE in every stage.

(c) The socially optimal level:

$Max_{q_1,q_2}(u_1 + u_2) = (-2q_1 + 3q_2) + (-2q_2 + 3q_1) = q_1 + q_2$
s.t.:$0 \leq q_i \leq 1$

The maximum is obtained at the corner solution $q_1 = q_2 = 1$. With utility 1 for each one of them.

Trigger strategy (TS): At $t = 0$: $q_i = 1$. At $t > 0$: $q_i = 1$ if $(1,1)$ in all past periods, $q_i = 0$ otherwise.

If they play according to the TS: $u_i(TS) = 1 + \delta + \delta^2 + \cdots = \frac{1}{1-\delta}$.
If a deviation to $q_i = 0$ (D) occurs: $u_i(D, TS) = 3 + 0 + 0 + \cdots = 3$.
TS is better if $\delta \geq \frac{2}{3}$.

Check that then TS is a SPNE:

1. It is a NE of the game as δ was computed for this to be the case.
2. It is a NE of the subgames after (1,1) in all past periods as these subgames are the same as the whole game and TS prescribes the same thing in these subgames as in the whole game.
3. It is a NE of the subgames after some deviation as TS requires to play the one-shot NE for all futures periods unconditionally.

II.4 Time for decision in Westeros' Riverlands: After the death of late king Robert Baratheon, the houses Frey and Tully have been asked by Robert's son Joffrey to pledge loyalty to him as the new king. However, there is another pretender, Stannis, the brother of the dead king, who accuses Joffrey of being an illegitimate son. Both Freys and Tullies now face a choice: They have to decide whether to back Joffrey, or to back the pretender Stannis. The decisions are simultaneous made by the Head of each House.

House Tully is divided between two types of people: those who think Joffrey is a legitimate son (Type 1, with probability q) and those who think he is not (Type 2, with a probability $1 - q$).

In the case the Head of Tully is of Type 1: If both houses back Joffrey, he will be crowned king and will offer them an equal reward of 20 gold dragons (gd), which would be more than the 10 gd offered by Stannis. If one house backs Stannis and its neighbor backs Joffrey, Joffrey wins anyway, and the family who did not support him would be beheaded, which causes them a disutility of -20 gd and the winner would be granted the lands of its local rival for total gain of 30 gd.

In the case the Head of Tully is of Type 2: If both houses back Joffrey, it causes a disutility of -5 gd for House Frey and -10 gd for House Tully. If both houses back Stannis, Frey gets 20 gd and Tully, 30 gd. If Frey House backs Joffrey and its neighbor backs Stannis, they get respectively 15 gd and 0 gd. In the reversed situation Frey gets 0 gd and Tully 10 gd.

(a) Give the Bayesian form of the game. (5 points)
(b) Calculate the Bayesian Nash equilibria of this game as a function of q. (8 points)

Now suppose that the Head of the House of Tully, who is of Type 1, can send a message to Frey, either revealing he is of Type 1 or lying, pretending to be of Type 2.

(c) If it is known that Frey will trust the message, what would the head of the House of Tully say? (4 points)
(d) Would the House of Frey actually trust the message? (3 points)

(a)

	TULLY (Type 1 with prob q)		
		Joffrey	Stannis
FREY	Joffrey	$\underline{20}, \underline{20}$	$\underline{30}$, -20
	Stannis	-20, $\underline{30}$	10, 10

	TULLY (Type 2 with prob 1- q)		
		Joffrey	Stannis
FREY	Joffrey	-5, -10	15, $\underline{0}$
	Stannis	$\underline{0}$, 10	$\underline{20}, \underline{30}$

(b) Tully's best response it Type 1: BR_T (J) = J, BR_T (S) = J;
Tully's best response it Type 2: BR_T (J) = S, BR_T (S) = S.

Compute Frey's best response:
$u_F(J) = 20q + 15(1 - q) = 15 + 5q$

$u_F(S) = -20q + 20(1 - q) = 20 - 40q$
J is best response if $q \geq \frac{1}{9}$, S is best response if $q \leq \frac{1}{9}$.
BNE = $\{((J, JS)$ if $q \geq \frac{1}{9}), ((S, JS)$, if $q \leq \frac{1}{9})\}$.

(c) If Tully pretends to be of Type 1: Frey believes Tully will play J and Frey will also play J. Tully is indeed of Type 1, so he will play J and receive 20.
If Tully pretends to be of Type 2: Frey believes Tully will play S and Frey will also play S. But Tully is of Type 1, so he will play J and receive 30.
Tully will say he is of Type 2.

(d) Frey will not trust the message. If he is of Type 1 he would pretend to be of Type 2 as seen in (c). If he is of Type 2 he will also pretend to be of Type 2: by pretending to be of Type 1 he pretends he is playing J, which makes Frey choose J. But he will choos S and get 0, less than 30 if he is trusted to be of Type 2.
This means both types want to pretend being of Type 2, so the message provides no information and will not be trusted by Frey.