Game Theory
Exam January 2013

Name Group:

You have two and a half hours to complete the exam.

I. Short questions (20 points).

I.1 The following two statements are either true or false. If the statement is true, give an explanation. If the statement is false, provide a counter-example.

a) Any SPNE is also a NE.
b) If a game has no proper subgames, then the set of SPNE and the set of NE coincide.

I.2 Two identical firms compete in quantity in a market. Explain the consequences of changing from the static game (Cournot) to the dynamic game (Stackelberg): who wins, who loses and why.

I.3 Define the elements of a Bayesian game and also the concept of Bayesian Nashs equilibrium.

I.4 In a finitely repeated game one can always find a SPNE in which the payoffs are Pareto optimal. Is that statement true? Why?

Solutions:

I.1 a) True: A SPNE is, by definition, a NE of the game that, in addition, it is also a NE in every subgame. (2,5 points)
b) True: From a), when there are no subgames, it is enough that a strategy profile is a NE to be a SPNE. (2,5 points)

I.2 In the cases shown in class, the leader gains and the follower loses in Stackelberg respect to the profits they would obtain in Cournot. The reason is that the reaction functions indicate that, as the rival's quantity increases, one must reduces its own. The leader moves first and can increase its production knowing that the rival will reduce his. (5 points)

I.3 Players set: N={1,2,…,n}
Types set: $T=\{T_1, T_2, \ldots T_n\}$, where T_i is the set of types of Player i.
Probabilities of types: For every type of every player one must specify the probability that he assigns to the types of the other players: $p_i(t_{-i}/t_i)$, where $t_{-i} \in T_{-i}$ and $t_i \in T_i$.
Action set for every type of every player: A_i.
Payoff function: $u_i(a_i, a_{-i};\ t_i, t_{-i})$, where $a_{-i} \in A_{-i}$ and $a_i \in A_i$, $t_{-i} \in T_{-i}$ and $t_i \in T_i$.
(1 point for each element)

I.4 False. If there is only one NE in the static game, the repetition of the game will only have the repetition of the NE in every subgame as the SPNE. The NE does not need to be Pareto optimal as we saw, for instance, in the prisoners' dilemma game. (5 points)

II. Problems. You must answer any four of the following problems. (20 points each).

II.1 USA intelligence suspects that North Korea is thinking about starting a program to develop nuclear missiles. This will dramatically affect the stability of the region. According to the testimony of fugitives that abandoned the country, life conditions are very poor. This makes USA authorities propose the following agreement to North Korea:

- USA will give a 25 billion dollars aid to North Korea if it abandons its nuclear program.
- The aid will be given once the agreement is signed, but North Korea promises to give it back of the USA finds proof that the nuclear program is still on (asume this is a binding agreement). Without the proof, North Korea will not give the money back.
- To get the proof, USA can make the International Atomic Energy Agency inspect North Korean installations. The cost of the inspection is 10 billions and will be paid by the USA.

The USA finds that the stability of the region is worth 90 billion dollars, whereas for North Korea, to give up its nuclear program has an estimated cost of 20 billion dollars.

a) Identify the set of strategies for each player, the payoffs and the normal form of the game.
b) Find the Nash equilibria of the game.
c) Which are the expected payoffs for each player in the equilibrium?

Suppose that USA changes the amount of the aid to North Korea from 25 to 30 billion dollars.

d) Find the expected utility for USA in the new equilibrium. Do you find a paradox when comparing with c)? Explain it.

a) (4 points) Players USA (1) and North Korea (2)
Strategies: $S_1 = \{I, NI\}$ I= Inspect, NI= Do not inspect
$S_2 = \{M,NM\}$ M=Mantain the program, NM= Do not Mantain the program

	M	NM
I	-10, 0	55, 5
NI	-25, 25	65, 5

b) (4 points) NE = {(4/5, 2/5)}

c) (4 points) $U_1(4/5, 2/5) = 29$, $U_2(4/5, 2/5) = 5$

d)

	M	NM
I	-10, 0	50, 10
NI	-30, 30	60, 10

NE = {(2/3, 1/3)}
$EU_1=30 > 29$ and $p = 2/3$ and $q = 1/3$.
(4 points)

Paradox: USA pays a higher aid, but receives a higher payoff in equilibrium. The reason os that the increase in the aid makes North Korea more willing to take it and not to cheat, which implies that USA needs to inspect less often. All these things increase USA's payoff and decreases its costs. (4 points)

II.2 Tarde and Mercadina are the only two supermarkets competing for costumers in Getafe. The demand for Tarde's products is given by $q_1(p_1, p_2) = 11 - p_1 + 0.5p_2$. The demand for Mercadina's products is given by $q_2(p_1, p_2) = 11 - p_2 + 0.5p_1$. Both supermarkets face a cost of supplying their products that amounts to $C(q) = 4q$.

a) Compute the Nash equilibrium when both supermarkets simultaneously set their prices. Calculate the resulting profits.

Suppose now that, before the choice of prices, Tarde can develop a new technology at the cost of 10 units, which will reduce its marginal cost to zero. However, Mercadina will immediately copy the new technology paying a fixed cost of 5.

b) Calculate the SPNE of the game. Will Tarde develop the new technology?

a) Tarde solves the following problem:

$max_{p_1}(11 - p_1 + 0.5p_2)(p_1 - 4)$
(2 points)

From the FOC we obtain the reaction function: $p_1 = \frac{15+0,5p_2}{2}$.

Similarly for Mercadina we obtain: $p_2 = \frac{15+0,5p_1}{2}$.
(4 points)
Solving the system formed by the two reaction functions we get $p_1 = p_2 = 10$.
(2 points)
Substituting in the profits functions: $\Pi_1 = \Pi_2 = 36$.
(2 points)

b) If Tarde does not develop the technology, the game in a) will follow.
(2 points)
If Tarde develops the technology, it will solve the following problem:

$max_{p_1}(11 - p_1 + 0.5p_2)p_1$
(1 point)

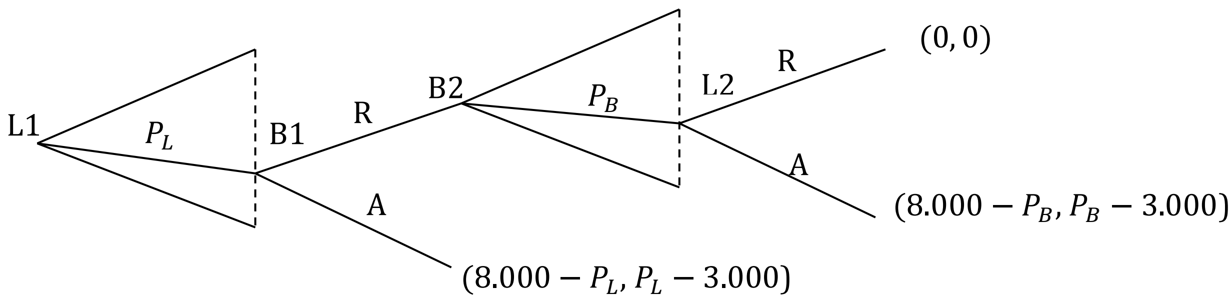
From the FOC we obtain the reaction function: $p_1 = \frac{11+0,5p_2}{2}$

Similarly for Mercadina we obtain: $p_2 = \frac{11+0,5p_1}{2}$.
(1 point)
Solving the system formed by the two reaction functions we get $p_1 = p_2 = \frac{22}{3} = 7{,}33$.
(1 point)
Substituting in the profits functions (do not forget to substract the fixed costs): $\Pi_1 = 43{,}72,\ \ \Pi_2 = 48{,}72$.
(1 point)
Despite the fact that Mercadina copies the technology and gets higher profits, Tarda is still better off if it develops the technology.
(1 point)
The SPNE is:
Tarda: (Develop the technology, $p_1 = 10\ if\ it\ does\ not\ develop\ the\ tech, p_1 = 7{,}33\ if\ it\ develops$).
Mercadina: ($p_2 = 10\ if\ Tarda\ does\ not\ develop\ the\ tech, p_2 = 7{,}33\ if\ it\ develops$).
(3 points).

II.3 Borja and Lorenzo are friends, and bargain over the sale of a 1000 cc motorcycle that Borja owns and Lorenzo is interested in. The value of the motorcycle for Borja is 3,000 euros, and for Lorenzo is 8.000 euros. They agree on the following bargaining procedure. First Lorenzo offers a price, then Borja accepts or rejects it. In case of accepting, the negotiation is over. If Borja does not accept Lorenzo's offer, Borja will offer a price that Lorenzo must accept or reject. In case of no agreement the procedure ends with no sale. In case a player is indifferent between accepting and rejecting he accepts. They both value future payoffs the same as present ones.

a) Represent the extensive form of the game and compute its SPNE.
b) What is the agreement they will reach, what are the payoffs, and in which period will the agreement be reached?
c) If the negotiations are initiated by Borja, who will benefit in the negotiation? What is the agreement they will reach, which are the payoffs, in which period will be reached?

a) Extensive form:



(5 points)
The SPNE is:
L2 accepts (A) if $8.000 - P_B \geq 0$, e.g., if $p_B \leq 8.000$; rejects otherwise.
B2 offers $p_B = 8.000$.
B1 accepts (A) if $P_L - 3.000 \geq (8.000 - 3.000)$, e.g., if $p_L \geq 8.000$.
L1 offers $p_L = 8.000$.
(5 points)

b) The agreement is reached in the first stage. Lorenzo (Buyer) offers $p_L = 8.000$ and Borja (Seller) accepts. Payoffs are $(8.000 - 8.000,\ 8.000 - 3.000) = (0, 5.000)$. Borja is better off.
(5 points)

c) The agreement is reached in the first stage. Borja (Seller) offers $p_B = 3.000$ and Lorenzo (Buyer) accepts. Payoffs are $(8.000 - 3.000, 3.000 - 3.000) = (5.000, 0)$. Lorenzo is better off.
(5 points)

II.4 In January 2013 two automobil firms must decide simultaneously whether to produce a new electric vehicle. Firm X will obtain 18 million euros in case it produces the electric vehicle as long as Firm Y does not produce his. On the other hand, if Firm Y produces his vehicle, and Firm X does not, Firm Y gains 15 million euros. In both cases, the firm that does not produce the vehicle gains just 7 million euros. If both of them produce the vehicle they get 8 million each. If no one produces the vehicle, they get 12 million each. All gains are per year.

a) If the decision has to be made every January 1st each of the following three years, and knowing that profits are increased by one million every year, find the SPNE and the payoffs in the SPNE.

Suppose now that the decision has to be made on January 1st every year indefinitely, and that profits each year are the same as the first year.

b) Will the firms be able to reach a cooperation agreement? Which one?
c) Compute the minimum discount factor for the firms to be willing to comply with the agreement.
d) Indicate the SPNE that supports the agreement. Compute the expected payoffs in that SPNE.

a) The static game in each stage is:

1ª etapa	S	NS
S	8, 8	18, 7
NS	7, 15	12, 12

2ª etapa	S	NS
S	9, 9	19, 8
NS	8, 16	13, 13

3ª etapa	S	NS
S	10, 10	20, 9
NS	9, 17	14, 14

NE in each static game is: {(S, S)}. (1 point)

Since there is only one NE in the static game, the only SPNE is the repetition of the NE: (1 point)
SPNE= {(Firm X: (1st stage: play S, 2nd stage: play S no matter what was played in 1st stage, 3rd stage: play S play S no matter what was played in previous stages), (Firm Y: (1st stage: play S, 2nd stage: play S no matter what was played in 1st stage, 3rd stage: play S play S no matter what was played in previous stages))}. (2 points)
Payoffs are: {(8+9+10 ; 8+9+10)}. (1 point)

b) Yes. They can reach an agreement not to produce the electric vehicle. The agreement could consist on playing the "trigger strategy": $S_X = S_Y =$ (play NS at $t = 1$, play NS at $t > 1$ if (NS,NS) was played in every $\tau < t$, play S at $t > 1$ if (NS,NS) was not played at some $\tau < t$).
(5 points)

c) Solving $u_x(S_X, S_Y) = \frac{12}{1-\delta_X} \geq 18 + \frac{8\delta_X}{1-\delta_X}$ we find $\delta_X \geq 3/5$ for Firm X to be willing to mantain the cooperation. Analogously for Y: $u_Y(S_X, S_Y) = \frac{12}{1-\delta_Y} \geq 15 + \frac{8\delta_Y}{1-\delta_Y}$ we find $\delta_Y \geq 3/7$ for Firm Y to be willing to mantain the cooperation. (5 points)

d) The "trigger strategy" (S_X, S_Y) described in b) with discount factors found in c) constitute a SPNE that gives payoffs of 12 in each period. (1 point)
If constitutes a NE in the whole game for $\delta_X \geq 3/5$ and $\delta_Y \geq 3/7$ as was calculated in c). Payoffs are, $u_x(S_X, S_Y) = \frac{12}{1-\frac{3}{5}} = 30$ and $u_Y(S_X, S_Y) = \frac{12}{1-\frac{3}{7}} = 21$. It is also OK to say that they get 12 per period (2 points).
It constitutes a NE in subgames in the subgames in which the strategy precribes to play (NS,NS) as the game is identical to the original game and because the trigger strategy (S_X, S_Y) prescribes the same actions as in the original game. (1 point)
It constitutes a NE in subgames in the subgames in which the strategy precribes to play (S,S) for ever as it is the repetition of the NE of the static game. (1 point)

II.5 Two firms, 1 and 2, compete a la Cournot. The (inverse) market demand function is $p = 6 - Q$, where p is the market price and Q is the total quantity produced by the two firms. Firm 2 has marginal costs $c_2 = 3$. Firm 1 has marginal cost equal to $c_A = \frac{4}{3}$ with probability 2/3 and to $c_A = \frac{10}{3}$ with probability 1/3. Firm 1 knows its marginal cost while Firm 2 only knows that it is c_A or c_B with probabilities described before.

a) Find the reaction functions of Firm 1.
b) Find the reaction function of Firm 2.
c) Find the Bayesian Nash Equilibrium.
d) Draw the reaction functions and find the equilibrium graphically using the fact, observed in the reaction function of Firm 2 seen in b), that Firm 2 reacts against the expected quantity of Firm 1.

a) Firm 1 may be of Type A (high costs, $c_A = \frac{10}{3}$) or Type B (low costs, $c_B = \frac{4}{3}$). Type A solves:
$max_{q_A}(6 - q_A - q_2)q_A - \frac{10}{3}q_A$, from which we obtain the reaction function: $q_A = \frac{6-q_2-\frac{10}{3}}{2}$.

Analogously for Type B:
$max_{q_B}(6 - q_B - q_2)q_B - \frac{10}{3}q_B$, from which we obtain the reaction function: $q_B = \frac{6-q_2-\frac{4}{3}}{2}$.
(5 points)

b) Firm 2 solves:

$max_{q_2}\frac{2}{3}[(6 - q_A - q_2)q_2 - 3q_2] + \frac{1}{3}[(6 - q_B - q_2)q_2 - 3q_2]$, from which we obtain the reaction function: $q_2 = \frac{6-(\frac{2}{3}q_B+\frac{1}{3}q_A)-3}{2}$.
(5 points)

c) Using the three reaction functions we find the BNE: $\{((q_B = 2, q_A = 1), (q_2 = \frac{2}{3}))\}$. (5 points)

d) $E(q_1) = \frac{2}{3}q_B + \frac{1}{3}q_A = \frac{4-q_2}{2}$. Hence: $q_2 = \frac{3-E(q_1)}{2}$.
The reaction functions can be drawn with $E(q_1)$ and q_2 in the axis. The equilibrium point can be represented as $E = (E(q_1) = \frac{2}{3}2 + \frac{1}{3}1 = \frac{5}{3}, q_2 = \frac{2}{3})$. (5 points)