I	II.1	II.2	II.3	II.4	Total

Game Theory
Exam December 2024

Name: **Reduced group:**

You have two and a half hours to complete the exam. No calculators or electronic devices are permitted. If you have a special need, please, contact the proctor.

I Short questions (5 points each)

I.1 Consider a market where two identical firms produce a homogenous good. Explain why the firms' profits are lower in equilibrium when they compete in prices (Bertrand) rather than when they compete in quantities (Cournot). Hint: base your argument on the best reply functions.

I.2 Provide an example of a game with a Nash equilibrium in which at least one player is playing a weakly dominated strategy.

I.3 Two players bargain over one euro with the rules seen in class. Offers must be an integer number of cents. Player 1 makes Player 2 a take-it-or-leave-it offer. Model this as a game and show the subgame perfect Nash equilibria. How many subgames are there?

I.4 In a negotiation game with uncertainty (about who makes the next offer, for instance), why would a player get a smaller share the more risk averse she is?

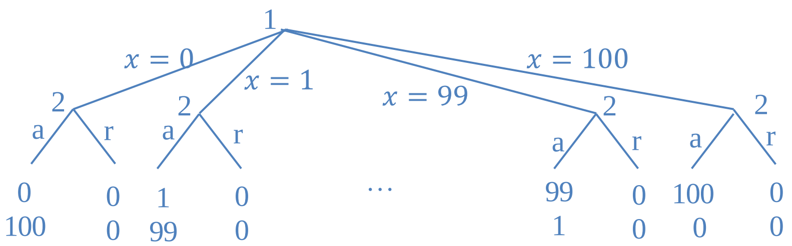
I.1 In the standard Bertrand, the strategies (prices) are strategic complements: the lower the rival's price, the lower price in the best reply (at least for the range of prices between marginal cost and monopoly price). This results in a fierce competition. In standard Cournot, quantities are strategic substitutes: the lower the quantity of the rival, the higher the quantity in the best reply. This mitigates competition.

I.2

		Player 2 A	Player 2 B
Player 1	A	5, 5	0, 4
	B	4, 0	0, 0

(B, B) is a NE, with both players playing a weakly dominated strategy.

I.3



$SPNE = \{(0, (\text{a } \forall \text{x})), (1, (\text{r if } x = 0, \text{a if } x > 0))\}$. There are 102 subgames.

I.4 If a player's payoffs in case she rejects are uncertain, the higher her risk aversion the lower share she will be willing to accept to avoid the uncertainty.

II. Problems (20 points each)

II.1 Two firms, *A* and *B*, compete *a la* Cournot in a market with demand $p = 60 - q$. The firms have no variable costs, but they both have a fixed cost equal to F. If a firm does not produce, it does not pay the fixed cost. Find the Nash-Cournot equilibria in pure strategies for each of the following cases. Hint: when looking for asymmetric equilibria, do not forget to check the best reply against a firm that is the only producer.

(a) (2 points) $900 < F$.
(b) (5 points) $400 < F < 900$.
(c) (5 points) $0 \leq F < 225$.
(d) (5 points + 3 extra points) $225 < F < 400$. Hint: look for equilibria like in (b) and (c).
(e) (3 points) $F = 900$, $F = 400$, $F = 225$. Hint: no need to do further calculations.

(a) The monopolistic quantity is found solving $\max_{q}(60 - q)q$, which gives $q^M = 30$. Revenues will be 900, smaller than the fixed cost. This means that there is no way to have positive profits in this market. The Nash-Cournot equilibrium is $(q_1 = 0, q_2 = 0\ )$.

(b) If both firms enter, Firm i solves $\max_{q_i}\left(60 - q_i - q_j\right)q_i$, with best reply after F.O.C.: $q_i = \frac{60-q_j}{2}$. Solving the system gives $q_1 = q_2 = 20$. Revenues are 400, lower than the fixed cost. There is no equilibrium with both firms producing.
If only one firm produces, say Firm *i*, it will produce the monopolistic quantity $q_i = 30$ with revenues 900, higher than the fixed cost. If Firm *j* produces, its best reply is $q_j = \frac{60-30}{2} = 15$, which will result in price $p = 60 - 30 - 15 = 15$ and revenues $R_j = 225$, lower than the fixed cost, so it prefers not to produce as the best reply. This is an equilibrium.

The two equilibria are $\{(q_1 = 30, q_2 = 0), (q_1 = 0, q_2 = 30)\}$.

(c) If both firms produce, they will choose $q_1 = q_2 = 20$, with revenues 400, higher than the fixed cost. This is an equilibrium.
If only one produces, say Firm *i*, it will choose $q_i = 30$ with revenues higher than the fixed cost. If Firm *j* produces, its best reply is $q_j = \frac{60-30}{2} = 15$, which will result in price $p = 60 - 30 - 15 = 15$ and revenues $R_j = 225$, higher than the fixed cost, so it prefers to produce as the best reply. Only one firm producing is not an equilibrium.

The only equilibrium is $(q_1 = 20, q_2 = 20)$.

(d) If both firms produce, they will choose $q_1 = q_2 = 20$, with revenues 400, higher than the fixed cost. This is an equilibrium.
If only one produces, say Firm *i*, it will choose $q_i = 30$ with revenues higher than the fixed cost. If Firm *j* produces, its best reply is $q_j = \frac{60-30}{2} = 15$, which will result in price $p = 60 - 30 - 15 = 15$ and revenues $R_j = 225$, lower than the fixed cost, so it prefers not to produce as the best reply. This is an equilibrium.

The three equilibria are $\{(q_1 = 20, q_2 = 20), (q_1 = 30, q_2 = 0), (q_1 = 0, q_2 = 30)\}$.

(e) $F = 900$: equilibria in (a) plus equilibria in (b): $\{(q_1 = 0, q_2 = 0\ ), (q_1 = 30, q_2 = 0), (q_1 = 0, q_2 = 30)\}$.
$F = 400$: equilibria in (b) plus (d): $\{(q_1 = 20, q_2 = 20), (q_1 = 30, q_2 = 0), (q_1 = 0, q_2 = 30)\}$.
$F = 225$: equilibria in (c) plus (d): $\{(q_1 = 20, q_2 = 20), (q_1 = 30, q_2 = 0), (q_1 = 0, q_2 = 30)\}$.

II.2 Two firms, 1 and 2, sell products that are imperfect substitutes of each other. They compete in prices. Firm $i = 1,2$ faces demand $q_i = 12 - p_i + p_j$ where $j = 1,2, j \neq i$. Assume zero marginal costs. Consider first that firms decide simultaneously.

(a) (5 points) Calculate the Nash equilibrium. Find equilibrium prices, quantities and profits.

Now Firm 1 (leader) chooses price first and, then, after observing its decision, Firm 2 (follower) makes its decision.

(b) (10 points) Calculate the subgame perfect Nash equilibrium.
(c) (3 points) Find equilibrium prices, quantities and profits. Compare between the leader and the follower.
(d) (2 points) Compare this equilibrium with that in part (a).

(a) Firm i solves $\max_{p_i}(12 - p_i + p_j)p_i$, which gives $p_i = \frac{12+p_j}{2}$ $(i, j = 1,2, j \neq i)$. The system gives the NE: $p_1 = p_2 = 12$. Quantities are: $q_1 = q_2 = 12$. Profits are $\Pi_i = 144$. Check S.O.C.

(b) Follower's best reply is as before: $p_2 = \frac{12+p_1}{2}$. Firm 1 solves

$$\max_{p_1}(12 - p_1 + p_2)p_1$$
$$\text{s.t.} \qquad p_2 = \frac{12+p_1}{2}.$$

The solution is $p_1 = 18$. SPNE $\left(p_1 = 18, p_2 = \frac{12+p_1}{2}\right)$.

(c) $p_1 = 18, p_2 = 15; q_1 = 9, q_2 = 15; \Pi_1 = 162, \Pi_2 = 225$. The follower has an advantage.

(d) Both firms are better off in the dynamic game.

II.3 Consider the following prisoner's dilemma game repeated infinitely many times, where $x \geq 6$ and both players have discount factor $\delta = 0.5$

		Player 2	
		T	H
Player 1	T	5, 5	0, x
	H	x, 0	2, 2

(a) (10 points) Find the values of x to sustain the payoffs (5, 5) in all periods as a subgame perfect Nash equilibrium using a trigger strategy.

(b) (10 points) Assume $x \geq 7.5$ and consider the following strategy profile: players start playing (T, T) and continue to play so if no one deviates. If any of them deviates, both will play H during 2 periods and then they will play T until any of them deviates again, where they will have to play again H during 2 periods and then again T and so on until infinite. Is this strategy a subgame perfect Nash equilibrium? *(Another way to describe the proposed strategy profile is as follows: the profile has normal and punishment phases. In normal phase, the play is (T, T), in punishment phases, it is (H, H). The strategy begins in a normal phase. Normal phases continue as long as no one deviates from (T, T). If a player deviates in a normal phase, the strategy profile goes to punishment phase for the following two periods, after that, it goes back to a normal phase regardless of what they do in the punishment phase.)*

(a) Trigger strategy profile: at $t = 1$ play (T, T); at $t > 1$ play (T, T) if (T, T) was played at all $t' > t$, play (H, H) otherwise.
For the strategy profile to be a SPNE we need to show:
-It is a NE of the whole game:
If players follow the trigger strategy: $u_i(TS_1, TS_2) = 5\frac{1}{1-0.5} = 10$.
If Player *i* deviates only in the first period (deviation D_i^1), then $u_i\left(D_i^1, TS_2\right) = x + 2\frac{0.5}{1-0.5} = x + 2$.
The deviation is not profitable if $10 \geq x + 2$, or $x \leq 8$.
If Player *i* deviates in more periods, payoffs are even lower, as every deviation after the first one gives a payoff of 0 rather than 2 in the deviating period with no changes in non-deviating periods.
-It is a NE in subgames after (T, T) in all periods before: same conditions as for the NE in the whole game (the subgame is the same as the whole game and the trigger strategy says the same thing.)
-It is a NE in other subgames: the trigger strategy requires the unconditional play of (H, H) in all periods. This is a NE as any deviation will produce 0 rather than 2 in the deviating period with no changes in non-deviating periods.

(b) The strategy profile (call it (s_1, s_2)), is not a SPNE. It is enough to find a profitable deviation for one player in one subgame.
If players follow the strategy: $u_i(s_1, s_2) = 5\frac{1}{1-0.5} = 5 + 5 \times 0.5 + 5 \times 0.5^2 + 5\frac{0.5^3}{1-0.5}$.
If Player *i* deviates only in the first period (deviation d_i^1), then $u_i\left(d_i^1, s_2\right) = x + 2 \times 0.5 + 2 \times 0.5^2 + 5\frac{0.5^3}{1-0.5}$.
Comparing utilities, the deviation is profitable if:
$x + 2 \times 0.5 + 2 \times 0.5^2 + 5\frac{0.5^3}{1-0.5} \geq 5 + 5 \times 0.5 + 5 \times 0.5^2 + 5\frac{0.5^3}{1-0.5}$ if
$x + 2 \times 0.5 + 2 \times 0.5^2 \geq 5 + 5 \times 0.5 + 5 \times 0.5^2$ or
$x + 1.5 \geq 8.75$ or $x \geq 7.25$. Since this is the case, the strategy profile is not a NE of the whole game, and *a fortiori*, is not a SPNE of the game.

II.4 In the railway market, the company RENFESA can choose between transforming its manufacturing plant or not transforming it, T or NT. The profits from the choice will depend on whether the competitor company, AVLOSA, innovates its high-speed trains or not, I or NI. AVLOSA knows the costs it would incur if it decides to innovate or not innovate its trains, which can be high or low, but RENFESA does not know with certainty the costs its competitor would face. The probability that the costs are high is p, where $0 < p < 1$. Knowing that the decisions to transform or not transform, innovate or not innovate, are made simultaneously, and that the payoffs faced by both companies are as follows:

High costs		AVLOSA	
		I	NI
RENFESA	T	4, 10	6, -5
	NT	0, 12	10, 10

Low costs		AVLOSA	
		I	NI
RENFESA	T	12, 10	-6, 5
	NT	10, 0	8, 10

(a) (5 points) Write the elements of this Bayesian game.
(b) (15 points) Find the pure strategies Bayesian Nash equilibria of the game for each value of p. Find also the corresponding equilibrium payoffs.

(a) Players: $\{R, A\}$. Types: $T_A = \{A_H, A_L\}$, $R = \{R\}$. Beliefs: $\{((p(R|A_H) = 1), ((p(R|A_L) = 1),$ $((p(A_H|R) = p, p(A_L|R) = 1 - p)\}$. Actions: $A_R = \{T, NT\}$, $A_{A_H} = A_{A_L} = \{I, NI\}$. Strategies: $S_R = \{T, NT\}$, $S_A = \{(I, I), (I, NI), (NI, I), (NI, NI)\}$. Payoffs as in the matrices.

(b) A_H has I as dominant action.

If R plays T, A's best reply is (I, I).
If A plays (I, I), R's best reply is calculated after:
$u_R(T, (I, I)) = 4p + 12(1 - p) = 12 - 8p$,
$u_R(NT, (I, I)) = 10(1 - p) = 10 - 10p$.
T is best reply if $12 - 8p \geq 10 - 10p$, which is satisfied for all $p \in [0,1]$.
Then, $(T, (I, I))$ is a BNE for all p.

If R plays NT, A's best reply is (I, NI).
If A plays (I, NI), R's best reply is calculated after:
$u_R(T, (I, NI)) = 4p - 6(1 - p) = 10p - 6$,
$u_R(NT, (I, NI)) = 8(1 - p) = 8 - 8p$.
NT is best reply if $8 - 8p \geq 10p - 6$, or $p \leq \frac{7}{9}$.
Then, $(NT, (I, NI))$ is a BNE for $p \leq \frac{7}{9}$.

Then, for $p \leq \frac{7}{9}$ both $(T, (I, I))$ and $(NT, (I, NI))$ are BNE.
For $p > \frac{7}{9}$ $(T, (I, I))$ is the only BNE.