I	II.1	II.2	II.3	II.4	Total

Game Theory
Exam December 2023

Name: **Group:**

You have two and a half hours to complete the exam. No calculators or electronic devices are permitted. If you have a special need, please, contact the proctor.

I Short questions (5 points each)

I.1 In a normal form game where mixed strategies are not allowed, the strategy profile $s = (s_1, s_2)$ is a Nash Equilibrium. If mixed strategies are allowed, s is still a Nash equilibrium. True or false?

I.2 Provide an example of a game with the following properties:

(i) There is only one equilibrium.
(ii) If a strategy of Player i is removed, then the new game has only one equilibrium, and Player i is better off in that equilibrium than in the equilibrium in (i).

I.3 Two players bargain over one euro with the rules seen in class. Player 1 makes Player 2 a take-it-or-leave-it offer. Model this as a game and show the subgame perfect Nash equilibria.

I.4 In a negotiation game with discount factor $\delta \in (0,1)$, when is it better to make the last offer? And the first offer?

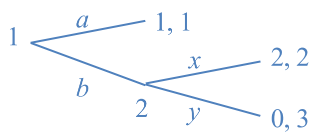
I.1 True. If a mixed strategy for Player i is a best reply against s_j and is better than s_i for Player i, then any pure strategy played with positive probability in that mixed strategy is also better than s_i. This contradicts the fact that s_i is a best reply against s_j.

I.2 An example in static games:

		Player 2	
		C	D
Player 1	C	4, 4	0, 3
	D	5, 0	1, 1

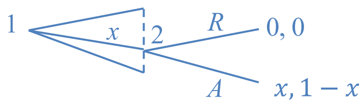
The Nash equilibrium is (D, D) with payoff for Player 1.
If D is eliminated for Player 1, the only equilibrium is (C, C), with payoff 4 for Player 1.

An example in dynamic games:



The subgame perfect Nash equilibrium is (a, y), with payoff 1 for Player 2.
If y is eliminated for Player 2, the only subgame perfect Nash equilibrium is (b, x), with playoff 2 for Player 2.

I.3



1 is Player 1, 2 is Player 2. Actions for Player 1 at his only information set are real numbers $x \in [0,1]$. Actions for Player 2 in his only information set are $A =$ accept, and $R =$ reject. Number x is the part of the euro corresponding to Player 1.

The only subgame perfect Nash equilibrium is $(1,\, a \text{ if } x \leq 1)$.

I.4 Best to be the last player making the offer: if the game has few rounds. This is because the other player will accept anything better than zero in the last round, giving Player 1 an advantage in the last and near-to-last rounds.
Best to be the first player making the offer: if the game has many rounds. This is because the first player can take advantage of the impatience of the other player. The advantage of the player making the last offer disappears as the last round is too far in the future.

II. Problems (20 points each)

II.1 Two supermarkets are located on the same street. They advertise their goods and by so doing attract customers to the area so that each of the two supermarkets' advertising is beneficial to both. Let x_1 be the advertising level of Supermarket 1 and x_2 be the advertising level of Supermarket 2. The profit functions for supermarkets 1 and 2 are respectively:

$$\pi_1(x_1, x_2) = (30 + x_2)x_1 - 2x_1^2$$
$$\pi_2(x_1, x_2) = (30 + x_1)x_2 - 2x_2^2$$

(a) (10 points) What are the Nash equilibrium advertising levels of the two supermarkets?
(b) (2 points) What are their equilibrium profits?
(c) (8 points) Could the supermarkets increase total profits by committing to given advertising levels? What advertising levels would they choose?

(a) Supermarket 1 solves: $\max_{x_1}(30 + x_2)x_1 - 2x_1^2$. F.O.C. give $x_1 = \frac{30+x_2}{4}$. Similarly, $x_2 = \frac{30+x_1}{4}$. (Check S.O.C.). The NE is: $(x_1 = 10, x_2 = 10\ )$.

(b) $\pi_1(x_1 = 10, x_2 = 10) = \pi_1(x_1 = 10, x_2 = 10) = 200$.

(c) Maximum total profits are found solving

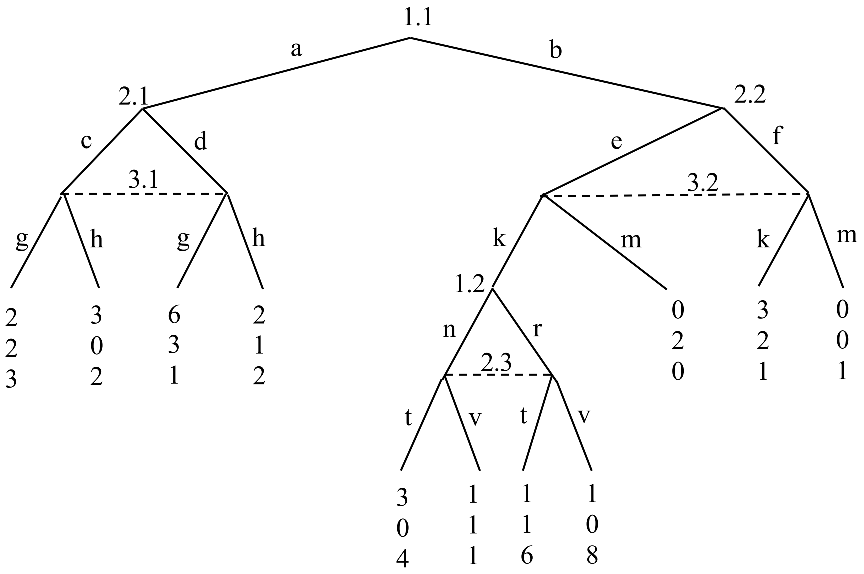
$$max_{x_1,x_2}\, \pi_1(x_1, x_2) + \pi_2(x_1, x_2) = max_{x_1,x_2}(30 + x_2)x_1 - 2x_1^2 + (30 + x_1)x_2 - 2x_2^2$$

F.O.C. give $30 + x_2 - 4x_1 + x_2 = 0$ and $30 + x_1 - 4x_2 + x_1 = 0$, with solution $x_1 = x_2 = 15$.

(Actually, other quantities like $15 \geq x_1 > 10$ and $15 \geq x_2 > 10$ also give higher total profits.)

Profits would be: $\pi_1(x_1 = 15, x_2 = 15) = \pi_1(x_1 = 15, x_2 = 15) = 225$.

II.2 Consider the following dynamic game for three players.



(a) (1 point) Identify the information sets for each player.
(b) (3 points) Show the strategy set for each player.
(c) (1 point) Find the subgames.
(d) (15 points) Find the subgame perfect Nash equilibria in pure strategies.

(a) Information sets for Player 1: {1.1, 1.2}, for Player 2: {2.1, 2.2, 2.3}, for Player 3: {3.1, 3.2}.

(b) $S_1 = \{(a, n), (a, r), (b, n), (b, r)\}$,
$S_2 = \{(c, e, t), (c, e, v), (c, f, n), (c, f, v), (d, e, t), (d, e, v), (d, f, n), (d, f, v)\}$,
$S_3 = \{(g, k), (g, m), (f, k), (f, m)\}$.

(c) There are 4 subgames, starting at 1.1, 2.1, 2.2 and 1.2.

(d) NE in pure strategies of the subgame starting at 2.1: $\{(d, h)\}$ with payoffs (2,1,2).
NE in pure strategies of the subgame starting at 1.2: (n, v), with payoffs (1,1,1).

Substitute the subgame at 1.2 with the equilibrium payoffs (1,1,1). Then, the NE in pure strategies of subgame 2.2 is (f, k), with payoffs (3,2,1).
Now substitute subgames 2.1 and 2.2 with their respective equilibrium payoffs, (2,1,2) and (3,2,1).
The NE in the subgame starting at 1.1 is {b}.
Then: $(b, (d, h), (f, k), (n, v))$ is a SPNE (with actions ordered by subgame. With actions ordered by player (the canonical way) we would write: $((b, n), (d, f, v), (h, k))$.

II.3 Consider the following prisoner's dilemma game repeated infinitely many times.

		Player 2	
		C	D
Player 1	C	4, 4	0, 6
	D	6, 0	1, 1

(a) (3 points) Define the trigger strategy that sustains the payoffs (4, 4) in all periods.
(b) (10 points) Show that it is a subgame perfect Nash equilibrium for some discount factor.
(c) (7 points) Consider now the strategy:

(i) Play (C, C) in period one and keep playing (C, C) unless in a punishment phase.
(ii) If a player deviates from the above at period t, a punishment phase starts that lasts two periods, $t+1$ and $t+2$. In this phase, play (D, D).

Show it is a subgame perfect Nash equilibrium for discount factor $\delta = 0.95$.
Note 1: You can use the approximations $0.95^2 \approx 0.9$ and $0.95^3 \approx 0.86$.
Note 2: notice that, after the punishment phase, (i) applies again.

(a) At $t = 1$ play (C, C).
At $t > 1$ play (C, C) if (C, C) was played at all $t' < t$. Otherwise play (D, D).

(b) It is a NE in the whole game:
If players follow the strategy, each one gets $u_i = \frac{4}{1-\delta}$.
If one deviates at $t = 1$, she gets: $u_i(\text{deviation at } t = 1) = 6 + \frac{\delta}{1-\delta}$.
The deviation is not profitable as long as $\frac{4}{1-\delta} \geq 6 + \frac{\delta}{1-\delta}$, which implies $\delta \geq 0.4$.

This deviation at $t = 1$ is the best possible deviation: any further deviation gives 0 rather than 1 with no future higher payoffs. Thus, the trigger strategy is a NE of the whole game.
It is also a NE of subgames after no deviations, as both the subgame and the strategy are the same as at the beginning of the game.
Finally, it is a NE of subgames after a deviation, as the strategy indicates an unconditional NE in all periods. Any deviation gives 0 rather than 1 in the deviating period with no future higher payoffs.

Thus, the trigger strategy is a SPNE as long as $\delta \geq 0.4$.

(c) It is a NE in the whole game:
If players follow the strategy, each one gets $u_i = \frac{4}{1-\delta}$.
If one deviates at $t = 1$, she gets: $u_i(\text{deviation at } t = 1) = 6 + \delta + \delta^2 + \frac{4\delta^3}{1-\delta}$.
The deviation is not profitable as long as $\frac{4}{1-\delta} \geq 6 + \delta + \delta^2 + \frac{4\delta^3}{1-\delta}$.
If $\delta = 0.95$, the inequality becomes $80 \geq 6 + 0.95 + 0.9025 + 80 \times 0.857375$, which is satisfied.

This deviation at $t = 1$ is the best possible deviation. Any deviation in the punishment phase gives 0 rather than 1 with no future higher payoffs. After the punishment phase, the game and the strategy are the same as at the beginning of the game. Thus, for $\delta = 0.95$ the strategy is also a NE in those subgames.

II.4 Consider a Cournot duopoly in a market with inverse demand function given by $p(q) = 120 - q$, where $q = q_1 + q_2$ is the market total quantity. Firm 1's total costs are $c_1(q_1) = 40q_1$ with probability 1/2, and $c_1(q_1) = 10q_1$ with probability 1/2. Firm 2's total costs are $c_2(q_2) = 20q_2$. Each firm knows its own costs. Firm 1 also knows Firm 2' costs, but Firm 2 does not know Firm 1's costs, although it knows Firm 1's possible types and probabilities. All this is common knowledge.

(a) (5 points) Show the Bayesian game, defining all its elements
(b) (15 points) Calculate the Bayesian-Nash equilibrium of the game. Calculate also the equilibrium prices and profits.

(a) Players: $N = \{1, 2\}$. 1 is Firm 1 and 2 is Firm 2.

Types: $T_1 = \{1.40, 1.10\}$, $T_{\text{James Dean}} = \{2\}$.

Beliefs: $\left(p(1.40 \mid 2) = \frac{1}{2}, p(1.10 \mid 2) = \frac{1}{2}\right)$,
$(p(2 \mid 1.40) = 1)$,
$(p(2 \mid 1.10) = 1)$.

Actions: $A_{1.40} = \{q_{1.40} \in [0, \infty)\}$, $A_{1.10} = \{ q_{1.10} \in [0, \infty)\}$, $A_2 = \{q_2 \in [0, \infty)\}$.

Strategies: $A_1 = \{(q_{1.40} \in [0, \infty),\ q_{1.10} \in [0, \infty))\}$, $S_2 = \{q_2 \in [0, \infty)\}$.

Payoffs:
For Type 1.40: $(120 - q_{1.40} - q_2 - 40)q_{1.40}$.
For Type 1.10: $(120 - q_{1.10} - q_2 - 10)q_{1.10}$.
For Type 2: $\frac{1}{2}(120 - q_{1.40} - q_2 - 20)q_2 + \frac{1}{2}(120 - q_{1.10} - q_2 - 20)q_2$.

For Player 1: $\frac{1}{2}(120 - q_{1.40} - q_2 - 40)q_{1.40} + \frac{1}{2}(120 - q_{1.10} - q_2 - 10)q_{1.10}$.
For Player 2: $\frac{1}{2}(120 - q_{1.40} - q_2 - 20)q_2 + \frac{1}{2}(120 - q_{1.10} - q_2 - 20)q_2$

(b) 1.40 solves: $\max_{q_{1.40}}(120 - q_{1.40} - q_2 - 40)\, q_{1.40}$. F.O.C. give the best reply: $q_{1.40} = \frac{80 - q_2}{2}$.

1.10 solves: $\max_{q_{1.10}}(120 - q_{1.40} - q_2 - 10)\, q_{1.10}$. F.O.C. give the best reply: $q_{1.10} = \frac{110 - q_2}{2}$.

2 solves $\max_{q_{1.10}} \frac{1}{2}(120 - q_{1.40} - q_2 - 20)\, q_2 + \frac{1}{2}(120 - q_{1.10} - q_2 - 20)q_2$.
F.O.C. give the best reply: $q_2 = \frac{1}{2}\frac{100 - q_{1.40}}{2} + \frac{1}{2}\frac{100 - q_{1.10}}{2}$.

The BNE is the solution of the system formed by the best reply functions:
BNE $= \left((q_{1.40} = 22.5, q_{1.10} = 37.5\,), q_2 = 35\,\right)$.

Prices are $p_{1.40} = 120 - 35 - 22.5 = 62.5$ when Firm 1 has high costs and $p_{1.10} = 120 - 35 - 37.5 = 47.5$ when Firm 1 has low costs.

Profits are: $\Pi_{1.40} = 22.5^2$, $\Pi_{1.10} = 37.5^2$ and $\Pi_2 = 35^2$.
Profits for Firm 1 are $\Pi_1 = \frac{1}{2}22.5^2 + \frac{1}{2}37.5^2$