I	II.1	II.2	II.3	II.4	Total

Game Theory
Exam December 2022

Name: **Group:**

You have two and a half hours to complete the exam. No calculators or electronic devices are permitted. If you have a special need, please, contact the proctor.

I Short questions (5 points each)

I.1 Find the set of rationalizable strategies in the following game:

		Microsoft		
		Cloud	Mobile	Quantum
	Cloud	2,4	4,2	1,3
Alphabet	Mobile	4,4	4,1	3,3
	Quantum	3,1	3,2	5,5

I.2 In a negotiation or bargaining game with alternating offers and no discount factor, the player making the last offer has all the negotiation power. If true, show why. If false, find a counterexample.

I.3 When do the definitions of "backward induction" and "subgame perfect Nash equilibrium" coincide? Explain.

I.4 Consider a duopoly with the two firms having equal, constant marginal costs and no fixed costs, facing a linear demand. A firm prefers the static Cournot outcome rather than being the leader in a Stackelberg scenario. True or false? Provide an example.

I.1 Step1: Mobile is dominated by Quantum for Microsoft. Stet 2: Cloud is dominated by either Mobile or Quantum for Alphabet. There are no further dominations. The set of rationalizable strategies is $\{$(Mobile, Quantum), (Cloud, Quantum)$\}$.

I.2 True. If the game reaches the last stage, she gets all the surplus, as the other player will accept anything. Anticipating that, by rejecting anything smaller than the whole surplus and by always offering to take it all, she guarantees that she will receive everything.

I.3 They coincide in perfect information games. The smallest subgames coincide with the play of the last player in that part of the game, and the NE coincides with the best action for that player. As we substitute these subgames with the corresponding payoff we have the same situation for the next smallest subgames.

I. 4 False. Demand: $Q = 12 - p$, zero costs. Cournot equilibrium: $q_i = 4$, equilibrium price: $p = 4$, profits: $\Pi_i = 16$. Stackelberg equilibrium: $\left(q_{\text{leader}} = 6, q_{\text{follower}} = \frac{12-q_{\text{leader}}}{2}\right)$, equilibrium quantities: $(q_{\text{leader}} = 6, q_{\text{follower}} = 3)$, equilibrium price: $p = 3$, profits: $\Pi_{\text{leader}} = 18$, $\Pi_{\text{follower}} = 9$. Firms prefer to be the leader in this example of the Stackelberg scenario.

II. Problems (20 points each)

II.1 Two friends, María and Ana, play the following game. María must choose a real number x in the interval $[0,2]$, and Ana another real number y in the interval $[0,1]$. María's utility function is $u_M(x,y) = 4\left(\frac{x}{2} - y\right)^2$, and Ana's is $u_A(x,y) = (x - 2y)^2$.

(a) (2 points) Assume that both friends choose their numbers simultaneously. Which are Ana's best replies for the values $x = 0, x = 1, x = 2$ chosen by María?

(b) (3 points) Which are María's best replies for the values $y = 0, y = 1/2, y = 1$ chosen by Ana?

(c) (5 points) Find both players' best replies.

(d) (7 points) Find the Nash equilibria.

(e) (3 points) Find the equilibrium payoffs.

(a) $BR_A(x = 0) = 1, BR_A(x = 1) = \{0,1\}, BR_A(x = 2) = 0.$

(b) $BR_M(y = 0) = 2, BR_M\left(y = \frac{1}{2}\right) = \{0,2\}, BR_M(y = 1) = 0.$

(c)

$$BR_M = \begin{cases} 2, \ if\ 0 \leq y < \frac{1}{2} \\ \{0,2\}, \ if\ y = \frac{1}{2} \\ 0, \ if\ \frac{1}{2} < y \leq 1 \end{cases}$$

$$BR_A = \begin{cases} 1, \ if\ 0 \leq x < 1 \\ \{0,1\}, \ if\ x = 1 \\ 0, \ if\ 1 < x \leq 2 \end{cases}$$

(d) There are two Nash equilibria: $\{(0, 1), (2,0)\}$. The first strategy is Maria's.

(e) Equilibrium payoffs are $u_M(0,1) = u_A(0,1) = 4;\ u_M(2,0) = u_A(2,0) = 4.$

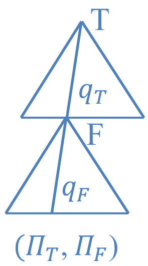
II.2 The train route Madrid – Barcelona has been attended by the company Trenfe so far, but the new operator FrenchCF is now entering the market. The two companies will compete as follows: the incumbent (Trenfe) will decide first the number of services they want to offer each week. The entrant (FrenchCF) will choose then their supply of services after observing Trenfe's decision. The incumbent has a cost function $C_T(q_T) = 40 + 15q_T$, while the entrant has $C_F(q_F) = 50 + 10q_F$. They both face the demand $Q(p) = 100 - 2\,p$, being q_T and q_F and Q the number of services per week by Trenfe, FrenchCF and total, respectively.

(a) (3 points) Represent the extensive form of the game.
(b) (7 points) Find the best response function for FrenchCF and the Subgame Perfect Nash Equilibrium.

Assume now that the game changes to a simultaneous competition.

(c) (5 points) Calculate the new best response functions of both players and the new Nash Equilibrium.
(d) (5 points) How much is Trenfe willing to pay to FrenchCF to go back to the game described in (a)? Would FrenchCF accept? Which game would the consumers prefer?

(a)



(b) From the demand: $p = 50 - \frac{q_T+q_F}{2}$.
FrenchCF's profit function is $\Pi_F = \left(50 - \frac{q_T+q_F}{2}\right) q_F - 50 - 10q_F$.
Maximization gives the best reply function: $q_F = 40 - \frac{q_T}{2}$.
Trenfe's profit function is $\Pi_T = \left(50 - \frac{q_T+40-\frac{q_T}{2}}{2}\right) q_T - 40 - 15q_T$.
Maximization gives $q_T = 30$
Subgame perfect Nash equilibrium is then: $\left(q_T = 30, q_F = 40 - \frac{q_T}{2}\right)$, with equilibrium path $q_T = 30, q_F = 25$.
Check that profits are positive: $\Pi_T = 185, \Pi_F = 262.5$

(c) FrenchCF's profit function is $\Pi_F = \left(50 - \frac{q_T+q_F}{2}\right) q_F - 50 - 10q_F$.
Maximization gives the best reply function: $q_F = 40 - \frac{q_T}{2}$.
Trenfe's profit function is $\Pi_T = \left(50 - \frac{q_T+q_F}{2}\right) q_T - 40 - 15q_T$.
Maximization gives the best reply function: $q_T = 35 - \frac{q_F}{2}$.
Solving the system, the Nash-Cournot equilibrium is $(q_T = 20, q_F = 30\,)$.
Profits are: $\Pi_T = 160, \Pi_F = 400$.

(d) Trenfe is willing to pay as much as $185 - 160 = 25$. FrenchCF will not accept less than $400 - 262.5 = 138.5$. FrenchCF will not accept. Consumers prefer the dynamic game, where the price is lower.

II.3 Consider the game that consists of repeating four times the next normal form game (assume there is no discounting):

		Player 2	
		S	P
Player 1	S	0, 0	5, 1
	P	1, 5	4, 4

(a) (2 points) What game of the ones seen in class is the most similar to this normal form game? Identify all pure strategy Nash equilibria. Is any of these equilibria efficient according to the utilitarian criterion (i.e.: the criterion that adds up the players' utilities)
(b) (2 points) How many subgames (including the whole game) has the described repeated game? How many information sets has each player? How many strategies?
(c) (3 points) Specify, as rigorously as possible, a subgame perfect Nash equilibrium (SPNE) where one of the players obtains an average payoff of 1 and the other player an average payoff of 5. The average is the total payoff divided by the number of periods.
(d) (3 points) Repeat (c), but for (average) payoffs of 3 for both players.
(e) (2 points) Is there any SPNE where both players obtain efficient average payoffs according to the utilitarian criterion (i.e., equal to 4)?
(f) (8 points) If the game is repeated infinitely many times with a discount factor of $\delta \in (0,1)$, could you sustain payoffs (4,4) in each period?

(a) It is a chicken game. NE in pure strategies: $\{(S,P),(P,S)\}$.
Sum of payoffs in both NEa are 6, while (P,P) gives 8. No NE is optimal in this sense.

(b) Subgames starting at $t=1$: 1. At $t=2$: 4. At $t=3$: 16, At $t=4$: 64. Total: 85.
For each player there is one information set per period per subgame starting in that period: 85.
A strategy for a player is a vector of 85 components, in each one there are two possibilities. The total number of strategies is 2^{85}.

(c) Play (S,P) at all stages. Same argument as in (b).

(d) At $t=1$ and $t=3$ play (S,P). At $t=2$ and $t=4$ play (P,S). As before, it is a SPNE as it indicates an unconditional NE of the stage game in every stage.

(e) No. In the last period the only possibilities are either (S,P) or (P,S), where one player must get a payoff of 1. Then it is impossible to get an average of 4.

(f) Use the trigger strategy: at $t=1$ play (P,P), $t>1$ play (P,P) if (P,P) was played at all $t'<t$; otherwise play (S,P) if Player 2 was the first player who did not play P, and (P,S) if Player 1 was the first player who did not play P. If both players deviated first, play any equilibrium.

Check it is a SPNE:
It is a NE of the whole game: $u_i(\text{trigger strategy})=\frac{4}{1-\delta}$.
Check one-period deviations (d^1): $u_1(d^1,\text{trigger strategy})=5+\frac{\delta}{1-\delta}$. Not worth if $5+\frac{\delta}{1-\delta}\leq\frac{4}{1-\delta}$, or $\delta\geq\frac{1}{4}$.
Deviations of more than one period are less profitable than d^1, as in all periods except the first one will give a payoff of 0 rather than 1.
It is a NE in subgames after a history of (P,P) in all periods: the subgame is the same as the whole game and the trigger strategy specifies the same play as in the whole game. The same argument as before ensures it is a NE in these subgames.
It is a NE in subgames after a history that includes any play other than (P,P) in all periods: The trigger strategy specifies playing an unconditional NE of the stage game in all stages. This is a NE of the repeated game.

II.4 Consider that the two firms in the market for frozen foods, FRISA and TORSA, behave as a Cournot duopoly, where the inverse demand function is given by $p(Q) = 30 - Q$, with $Q = q_f + q_t$ being the total market quantity. FRISA has a constant marginal cost equal to 6 with probability 1/4, and equal to 2 with probability 3/4. TORSA has costs given by $\cancel{C(q_t) = (q_t)^2}$, $C(q_t) = 3q_t$. FRISA knows its own costs and TORSA's, while TORSA knows its own costs, but not FRISA's, although it knows its possible costs and probabilities.

(a) (5 points) Indicate the elements of this Bayesian game.
(b) (10 points) Find the Bayesian Nash equilibria.

Consider now that firms compete in a Cournot duopoly where FRISA has a marginal cost equal to the expected cost in the above description.

(c) (5 points) Which are now the equilibrium quantities ~~profits~~ for both firms? Compare with the results obtained in (b).

(a) Players: {FRISA, TORSA}
Types: $T_F = \{f_1, f_2\}, T_T = \{t\}$.
Beliefs: $p(f_1|t) = 1/4, p(f_1|t) = 3/4; p(t|f_1) = 1; p(t|f_2) = 1$.
Actions for types: $A_f = \{q_1 \geq 0\}, A_{f_2} = \{q_2 \geq 0\}, A_t = \{q_t \geq 0\}$.
Utilities: $\Pi_1 = (30 - q_1 - q_t)q_1 - 6q_1$;
$\Pi_2 = (30 - q_2 - q_t)q_2 - 2q_2$;
$\Pi_t = \frac{1}{4}[(30 - q_1 - q_t)q_t - 3q_t] + \frac{3}{4}[(30 - q_2 - q_t)q_t - 3q_t]$.

(b) Maximizing the respective profit functions, obtain each type´s best reply function:
$q_1 = 12 - \frac{1}{2}q_t$,
$q_2 = 14 - \frac{1}{2}q_t$,
$q_t = \frac{27}{2} - \frac{1}{8}q_1 - \frac{3}{8}q_2$.
Solve the system to find the BNE: $\big((q_1 = 7.5, q_2 = 9.5), q_t = 9\big)$.

(c) New profit functions: $\Pi_f = \big(30 - q_f - q_t\big)q_f - 3q_f, \Pi_t = \big(30 - q_f - q_t\big)q_t - 3q_t$.
New reaction functions: $q_f = \frac{27-q_t}{2}, q_t = \frac{27-q_f}{2}$. NE: $q_f = q_t = 9$.
q_t is the same in both cases, q_f is the weighted average of q_1 and q_2: $\frac{1}{4}7.5 + \frac{3}{4}9.5 = 9$.