I	II.1	II.2	II.3	II.4	Total

Game Theory
Exam December 2021

Name: **Group:**

You have two and a half hours to complete the exam. No calculators or electronic devices are permitted.

I Short questions (5 points each)

I.1 Game theory predicts that players will always have a dominant strategy. If true, explain why; if false, provide a counterexample.

False. Counterexample: any game without dominant strategies, say the matching pennies. Even if we understand the equilibrium as the prediction, it is not a dominant strategy. Actually, whatever the players play it will never be a dominant strategy as the matching pennies has no dominant strategy.

I.2 In a finitely repeated game, there can be a SPNE in which, in the equilibrium path, in one of the periods, players may obtain payoffs above the ones in any of the Nash equilibria of the stage game only if the stage game has more than one Nash equilibrium. True or false? Explain

True. If there is only a NE in the stage game, its unconditional repetition is the only SPNE of the finitely repeated game. To get any other payoff it is a necessary condition (not sufficient) that the stage game has more than one NE.

I.3 Give an example of a game where there is a Nash equilibrium that is not subgame perfect.

See class notes.

I.4 Give an example of a game where a player wins more if she can eliminate one of her strategies compared to the situation in which she does not eliminate it.

See class notes (e.g.: Hernán Cortés game).

II. Problems (20 points each)

II.1 Consider three firms competing *a la* Cournot in a market with inverse demand function given by $P(Q) = 1 - Q$ and with production costs normalized to zero.

(a) (5 points) Find the Nash equilibrium of the game. Determine also the equilibrium price and profit levels for the firms.
(b) (10 points) Consider now that two firms merge to form a new one. Find the Nash equilibrium in this new game. Identify the equilibrium price and profits for the firms, and compare them with your pre-merger results in part (a).
(c) (5 points) Repeat (b) if the non-merging firm has a capacity constraint of 0.3.

(a) $\max_{q_1 \geq 0} \ (1-(q_1+q_2+q_3))q_1$ gives

$q_1 = BR_1(q_2, q_3) = max\left\{0, \ \frac{1-q_2-q_3}{2}\right\}$.

Similarly for firms 2 and 3:

$q_2 = BR_2(q_1, q_3) = max\left\{0, \ \frac{1-q_1-q_3}{2}\right\}$.

$q_3 = BR_3(q_1, q_2) = max\left\{0, \ \frac{1-q_1-q_2}{2}\right\}$.

Solving the system gives: $q_1 = q_2 = q_3 = \frac{1}{4}$. The Nash-Cournot equilibrium is $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$.

From that: $p = \frac{1}{4}$, and $\Pi_i = \left(\frac{1}{4}\right)^2$ for $i = 1, 2, 3$.

(b) 1 and 2 form firm 12

$\max_{q_{12} \geq 0} \ (1-(q_{12}+q_3))q_{12}$ gives

$q_{12} = BR_{12}(q_3) = max\left\{0, \ \frac{1-q_3}{2}\right\}$.

Similarly for firm 3:

$q_3 = BR_3(q_3) = max\left\{0, \ \frac{1-q_{12}}{2}\right\}$.

Solving the system gives: $q_{12} = q_3 = \frac{1}{3}$. The Nash-Cournot equilibrium is $\left(\frac{1}{3}, \frac{1}{3}\right)$.

From that: $p = \frac{1}{3}$, and $\Pi_i = \left(\frac{1}{3}\right)^2 = 0.111$ for $i = 12, 3$.

(c) If Firm 3 has a capacity constraint of 0.3, that is its production, as $0.3 < \frac{1}{3}$ Firm 12 will play the best reply against that: $q_{12} = BR_{12}(q_3 = 0.3) = \frac{1-0.3}{2} = 0.35$.

After that: $p = 1 - 0.35 - 0.3 = 0.35$. $\Pi_{12} = 0.35 \times 0.35 = 0.1225$, $\Pi_3 = 0.35 \times 0.3 = 0.105$.

II.2 Consider a 2-period bargaining game where two players can share a surplus of 10 euros. In the first period, Player 1 offers Player 2 that they divide the money as $(x, 10 - x)$, where the first component is the share of Player 1, and the second is the share of Player 2. Then, Player 2 chooses between accept and reject. If Player 2 accepts the game ends and both get the payoffs as proposed in $(x, 10 - x)$. If Player 2 rejects, the game comes to the second period, where both have to make simultaneously an offer. If Player 1 proposes $(y, 10 - y)$ and Player 2 proposes $(10 - z, z)$, payoffs are (y, z) if $y + z \leq 10$ and (0,0) otherwise. (I.e., if Players 1 claims y, and Player 2 claims z, they get what they claim as long as the sum of the claims do not exceed the 10 euros of the surplus.) The discount factor for both players is $\delta = 1/2$.

(a) (10 points) Solve the subgame that starts when Player 2 rejects the offer: Find best reply function for each player and, then, find the Nash equilibria of the subgame. You only need to consider strategies with $x, y, z \in [0,10]$.

(b) (10 points) Find all the subgame perfect Nash equilibria. Hint: You found many equilibria in (a), so you must find a subgame perfect Nash equilibrium for each one of them.

Note: If Player 2 is indifferent between accepting and rejecting, assume that he accepts.

(a) $BR_1(z) = y = 10 - z$, $BR_2(y) = z = 10 - y$.

From there:

$NE = \{((y, 10 - y), (10 - z, z))$ such that $y + z = 10; ((10, 0), (0, 10))\}$.

(b) For equilibria $((y, 10 - y), (10 - z, z))$ such that $y + z = 10$ in stage 2, in stage 1 Player 2 will accept any offer such that $10 - x \geq \frac{1}{2}z$. Player 1 proposes $\left(10 - \frac{1}{2}z,\ \frac{1}{2}z\right)$.

For equilibrium $((10, 0), (0, 10))$ in stage 2, in stage 1 Player 2 accepts any offer such that $10 - x \geq 0$, thus Player 1 will propose $(10, 0)$.

II.3 Consider the infinite repetition of the following prisoners dilemma:

		Player 2	
		C	D
Player 1	C	4, 4	0, 5
	D	5, 0	1, 1

(a) (5 points) Describe the trigger strategy to sustain (C, C) in all periods.
(b) (5 points) Describe a strategy similar to the trigger strategy, but where the consequence of no cooperation is not to play (D, D) for ever after, but to play (D, D) for *k* periods.
(c) (10 points) Find for which discount factor it is enough that the punishment lasts just one period ($k = 1$) for the strategy in (b) to be a subgame perfect Nash equilibrium.

(a) See class notes.

(b) At $t = 1$, play (C, C).
At $t > 1$:

- play (C, C) if (C, C) was played at $t - 1$, or if $t > k + 1$ and (D, D) was played at $t - 1$, $t - 2, \ldots, t - k$.
- play (D, D) otherwise.

(c) The utility of following the above strategy by both players is

$$u_i = 4 + 4\delta + 4(\delta)^2 + 4(\delta)^3 + 4(\delta)^4 + \cdots \text{ (calculated as usual).}$$

The utility at the time of deviation after deviating just once:

$$u_i = 5 + \delta + 4(\delta)^2 + 4(\delta)^3 + 4(\delta)^4 + \cdots$$

Therefore, this deviation is not profitable as long as $4 + 4\delta \geq 5 + \delta$, or $\delta \geq \frac{1}{3}$.
Deviating more than one period only continues the play of (D, D) and gets to (C, C) even later.
Deviating in subgames when (D, D) is supposed to be played, also delays the play of (C, C).

II.4 Two neighbors are planning the construction of a paddle court at a cost of 30 units. It is well known that Neighbor 1 values the court at 40 units, while the value for Neighbor 2 is her private information, although it is also known that the value can be either 40 (with probability p) or 20. They agree on the following decision mechanism. Each one sends a closed envelope to a mediator with their decision in favor or against making the court. If they both are in favor, they share the cost equally; if only one is in favor, she pays all the cost; finally, if both are against, the court is not made.

(a) (5 points) Describe the elements of the Bayesian game.
(b) (10 points) Calculate the Bayesian Nash equilibria of the game if $p = 1/5$.
(c) (5 points) Calculate the Bayesian Nash equilibria of the game if $p = 1/2$.

(a) Players: $N = \{1, 2\}$, where 1 is Neighbor 1 and 2 is Neighbor 2.
Types: $T_1 = \{1.40\}, T_2 = \{2.40, 2.20\}$
Probabilities:

$p(2.40|1.40) = p$, $(2.20|1.40) = 1 - p$
$p(1.40|2.40) = 1$,
$p(1.40|2.20) = 1$.

Actions: $A_{1.40} = A_{2.40} = A_{2.20} = \{F, A\}$, where F is "vote in favor", and A is "vote against".
Strategies: $S_{1.40} = \{F, A\}$, $S_2 = \{FF, FA, AF, AA\}$.
Payoffs:

p — 2.40

1.40	F	A
F	25, 25	10, 40
A	40, 10	0, 0

$1 - p$ — 2.20

1.40	F	A
F	25, 5	10, 20
A	40, -10	0, 0

(b) Try a BNE with 1.40 playing F:
$BR_{2.4}(F) = (A, A)$.
Calculate $BR_{1.4}(AA)$: $u_{1.40}(F, AA) = 10 > u_{1.40}(A, AA) = 0$. Then $BR_{1.4}(AA) = F$, and (F, AA) is a BNE.

Try a BNE with 1.40 playing A:
$BR_{2.4}(A) = (F, A)$.
Calculate $BR_{1.4}(FA)$:

$$u_{1.40}(F, FA) = \frac{1}{5} \times 25 + \frac{4}{5} \times 10 = 13 > u_{1.40}(A, FA) = \frac{1}{5} \times 40 + \frac{4}{5} \times 0 = 8.$$

Then $BR_{1.4}(FA) = F$ and there is no BNE with 1.40 playing A.

(c) Try a BNE with 1.40 playing F:
$BR_{2.4}(F) = (A, A)$.
Calculate $BR_{1.4}(AA)$: $u_{1.40}(F, AA) = 10 > u_{1.40}(A, AA) = 0$. Then $BR_{1.4}(AA) = F$, and (F, AA) is a BNE.

Try a BNE with 1.40 playing A:
$BR_{2.4}(A) = (F, A)$.
Calculate $BR_{1.4}(FA)$:

$$u_{1.40}(F, FA) = \frac{1}{2} \times 25 + \frac{1}{2} \times 10 = 17.5 < u_{1.40}(A, FA) = \frac{1}{2} \times 40 + \frac{1}{2} \times 0 = 20.$$

Then $BR_{1.4}(FA) = A$ and (A, FA) is a BNE.