## Game Theory <br> Exam December 2018

## Name:

## Group:

## You have two and a half hours to complete the exam. No calculators.

## I. Short questions ( 5 points each)

I. 1 Can there be a static game in which every strategy is rationalizable? If yes, provide an example. If no, explain why.

Yes, e.g. matching pennies.
I. 2 Represent in a table a game of the "prisoners' dilemma" type and, in another one, a game of the "battle of the sexes" type.

See class notes.
I. 3 In any dynamic game using backward induction yields a Nash equilibrium. Prove the assertion if it is correct or give a counterexample if it is not.

Not correct. In dynamic games with imperfect, asymmetric information, we cannot use backward induction.
I. 4 In a given game there are two Nash equilibria: $(a, a)$ and $(b, b)$. If the game is repeated three times, show that playing (a, a) during the first period, ( $\mathrm{b}, \mathrm{b}$ ) during the second period in all subgames and $(a, a)$ in the third period also in all is a SPNE.

In subgames starting the third period no one deviates, as the strategy prescribes a Nash equilibrium in each one of them.

In subgames starting the second period no one deviates in the second period as it will not improve payoffs in the second period (the strategy prescribes a NE), and will not affect the payoffs of the third period (in the third period the strategy does not depend on the actions of the previous period.)

In the whole game starting the first period the argument is the same as before.

## II. Problems (20 points each)

II. 1 Two division managers, labelled 1 and 2, can invest their effort in creating a better working relationship that benefits both, but the effort has a private cost. In particular, the payoff function for manager $i(i \in\{1,2\})$ from effort levels $\left(e_{i}, e_{j}\right)(i \in\{1,2\}, j \neq i)$ is:

$$
v_{i}\left(e_{i}, e_{j}\right)=\left(a+e_{j}\right) e_{i}-e_{i}^{2}
$$

where $a>0$.
(a) (5 points) What is the best response correspondence of each manager?
(b) (5 points) Are the two effort levels strategic complements or substitutes?
(c) (5 points) Find the Nash equilibria of this game.
(d) (5 points) Is there any efficient Nash equilibrium?
(a) $\frac{\partial v_{i}}{\partial e_{i}}=\left(a+e_{j}\right)-2 e_{i}=0$, from where $e_{i}=\frac{a+e_{j}}{2}$.
(b) Complements: $\frac{\partial e_{i}}{\partial e_{j}}=\frac{1}{2}>0$.
(c) Solving the system $\left(e_{1}=\frac{a+e_{2}}{2}, e_{2}=\frac{a+e_{1}}{2}\right)$ we find $e_{1}=e_{2}=a . N E=\{(a, a)\}$.
(d) There is only one NE and is not efficient:
$v_{i}(a, a)=a^{2}$,
If $e_{1}=e_{2}=2 a$, utilities are higher:
$v_{i}(2 a, 2 a)=2 a^{2}$.
II. 2 Consider the following game between two investors. They have each deposited 1000 euros in an investment fund managed by a bank. The bank has invested these deposits in long-term projects that last two periods. If any of the investors withdraws their money in the first period, the bank will be forced to liquidate its investment, and then the total return will be only 1600 euros, less than the total invested. If the project is completed, the total return is 2400 euros to be divided between the investors. Investors can make their withdrawal from the bank at the end of only one period or not at all. The decisions to withdraw or not in each stage are made simultaneously by the two investors. If both withdraw at stage 1 , then each gets 800 euros. If one withdraws at stage 1 and the other does not, the first one gets 1000 while the other gets 600 . If both withdraw at stage 2 , then each gets 1200 euros. If one withdraws at stage 2 and the other does not, the first one gets 1400 while the other gets 1000 . If no one withdraws, the bank gives 1200 back to each investor.
(a) (5 points) Represent the game in extensive form.
(b) (15 points) Find the pure-strategy Subgame Perfect Nash equilibria of the game.
(a)

(b) The only NE in subgame starting at 1.2 is ( $w, w$ ) with payoffs $(1200,1200)$. Replace the sugames with these payoffs to find the SPNEa of the whole game:
$S P N E=\left\{((W, w),(W, w)),((N, w),(N, w)),\left(\left(\frac{1}{2}[W]+\frac{1}{2}[N], w\right),\left(\frac{1}{2}[W]+\frac{1}{2}[N], w\right)\right)\right\}$.
II. 3 Firms 1 and 2 play a Cournot duopoly game. Market demand is $p=220-q_{1}-q_{2}$, were $p$ is the market price and $q_{i}$ is the quantity produced by Firm $i$. Marginal cost is 40 and there are no fixed costs for either firm. This market opens repeatedly, with a probability $\frac{8}{9}$ in each period that the market goes to the next period (and a probability $\frac{1}{9}$ that is no longer repeated). The discount factor is $\frac{9}{10}$.
(a) (2 points) Find the Cournot equilibrium for the game without repetition.
(b) (2 points) Find the monopoly quantity for the game without repetition.
(c) (2 points) Show that profits are bigger if each firm produces half the monopoly quantity than if each produces the Cournot quantity.
(d) (4 points) Show the trigger strategy that sustains the monopoly outcome with each firm producing half of it.
(e) (10 points) Show that the trigger strategy in (a) is a subgame perfect Nash equilibrium.
(a) After differentiating profits, reaction functions are given by $q_{i}=\frac{180-q_{j}}{2}$, which gives $q_{1}^{C}=$ $q_{2}^{C}=60$.
(b) $q^{M}=90$.
(c) $\Pi_{i}(45,45)=(220-45-45) \times 45-30 \times 45=4050$, $\Pi_{i}(60,60)=(220-60-60) \times 60-30 \times 60=3600$.
(d) Period 1: Play $q_{i}=45$,
period $t>1$ : Play $q_{i}=45$ if in all previous period players played $q_{1}=q_{2}=45$; if not, play $q_{i}=60$.
(e) If the strategy is followed, profits are:
$\Pi_{i}($ Trigger, Trigger $)=4050 \times \frac{1}{1-\frac{8}{9} \times \frac{9}{10}}=4050 \times \frac{1}{0.2}=20,250$.
If the Cournot quantity is produced in every period, profits are:
$\Pi_{i}($ Cournot, Cournot $)=3600 \times \frac{1}{1-\frac{8}{9} \times \frac{9}{10}}=3600 \times \frac{1}{0.2}=18,000$
The best deviation in one period is to produce $q_{i}=\frac{180-45}{2}=67.5$, with one-period profits
$\Pi_{i}(67.5,45)=(220-67.5-45) \times 67.5-30 \times 67.5=4,556.25$.
Total profits are:
$\Pi_{i}($ Deviation, Trigger $)=4,556.25+3600 \times \frac{\frac{8}{9} \times \frac{9}{10}}{1-\frac{8}{9} \times \frac{9}{10}}=4,556.25+14,400=18,956.25$, less than following the trigger strategy.

Now we can prove:

The trigger strategy is a NE in the whole game: as we saw, one-period deviations are not profitable. A deviation in more periods are deviations from the Cournot behavior, that is a NE in each period, which means that there is no improvement in the payoff of the period in which the firm deviates a second time, and there are also no effects of posterior periods, as the Cournot behavior is prescribed for all subsequent subgames, regardless of the past.

The trigger strategy is a NE in subgames after cooperation: both the game and the prescription by the trigger strategy are the same as in the whole game. The same argument applies.

The trigger strategy is a NE in subgames after some period with no cooperation: the trigger strategy prescribes the repetition of the one-shot NE, which is also a NE (in fact it is a SPNE)
II. 4 Consider the following games in normal form:

Player 2

|  | A | B |
| :---: | :---: | :---: |
| C | 1,2 | 0,0 |
| D | 0,0 | 2,1 |
|  |  |  |

(a)

Player 1
Player 2
C

(b)

Player 1 is informed about which of the two games is played. Player 2 only knows that game (a) is played with probability $0 \leq p \leq 1$ and game (b) is played with probability $1-p$. All this is common knowledge.
(a) (5 points) Describe the situation as a Bayesian game.
(b) (15 points) Find the Bayesian equilibria in pure strategies when $0<p<1 / 2$. (You will get only 7 points if you solve this question for a specific value of $p$ within the range.)
(a) $N=\{1,2\}$
$T_{1}=\left\{t_{a}, t_{b}\right\}, T_{2}=\{2\}$.
$\left(p\left(2 \mid t_{a}\right)=1\right),\left(p\left(2 \mid t_{b}\right)=1\right),\left(p\left(t_{a} \mid 2\right)=p, p\left(t_{b} \mid 2\right)=1-p\right)$.
$A_{t_{a}}=\{C, D\}, A_{t_{b}}=\{C, D\}, A_{2}=\{A, B\}$.
$S_{1}=\{C C, C D, D C, D D\}, S_{2}=\{A, B\}$.
Utilities as shown in the games (a) and (b).
(b) Consider the game of expected payoffs Player 2:

Player 2


If $0<p<1 / 2$ we have that $2 p<2(1-p)$ and $1-p>p$. The best replies of the players are shown in the game. So, the BNE are (CC, A), and (DD, B).

