Bayesian games

2. Economic applications

Universidad Carlos III de Madrid

Economic applications

- Auctions with asymmetric information over valuations:
  - First price auctions.
  - Second price auctions.
- Cournot duopoly with asymmetric information over:
  - Rival’s costs.
  - Demand.
- Public goods with asymmetric information over the rival’s valuation or costs.

Auctions

- Two individuals participate in a closed-envelope auction to buy an object.
- They must choose their bids, b_1 and b_2, simultaneously, and the object is sold to the one with the highest bid. In case of a tie, the winner is selected randomly.
- Valuations are v_1 and v_2.
- Player i's utility is:

$$u_i(b_1, b_2; v_1, v_2) = \begin{matrix} v_i - p & \text{if } b_i > b_j \\ \frac{v_i - p}{2} & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{matrix}$$

where p depends on the auction type used to sell the item.

First price auction

- The individual placing the highest bid gets the item. She pays her bid.
- Players: buyers 1 and 2.
- Possible actions: $b_i \in [0, \infty)$, $i = 1, 2$.
- Types: each buyer has a valuation v_i, that may be their private information.
- Player i's utility:

$$u_i(b_1, b_2; v_1, v_2) = \begin{matrix} v_i - b_i & \text{if } b_i > b_j \\ \dfrac{v_i - b_i}{2} & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{matrix}$$

First price auction

Let us see a simplified version of a first price auction:

- Buyer 1 may be of two types, with equal probabilities, one type values the item at 60 and the other values it at 100.
- Buyer 2 has only one type, that values the item at 60.
- Only bids allowed are 40, 60 and 80.
- We will not consider equilibria in weakly dominated strategies.

First price auction

$v_1 = 60$, prob $= 0.5$

Player 2

		$b_2 = 40$	$b_2 = 60$	$b_2 = 80$
	$b_1 = 40$	10, 10	0, 0	0, -20
Player 1.60	~~$b_1 = 60$~~	~~0, 0~~	~~0, 0~~	~~0, -20~~
	~~$b_1 = 80$~~	~~-20, 0~~	~~-20, 0~~	~~-10, -10~~

$v_1 = 100$, prob $= 0.5$

Player 2

		$b_2 = 40$	$b_2 = 60$	$b_2 = 80$
	~~$b_1 = 40$~~	~~30, 10~~	~~0, 0~~	~~0, -20~~
Player 1.100	$b_1 = 60$	40, 0	20, 0	0, -20
	$b_1 = 80$	20, 0	20, 0	10, -10

Eliminate weakly dominated strategies for 1.60 and 1.100.

First price auction

$v_1 = 60$, prob $= 0{,}5$

		Player 2		
		$b_2 = 40$	~~$b_2 = 60$~~	~~$b_2 = 80$~~
	$b_1 = 40$	10, 10	~~0, 0~~	~~0, -20~~
Player 1.60	$b_1 = 60$	0, 0	~~0, 0~~	~~0, -20~~
	$b_1 = 80$	-20, 0	~~-20, 0~~	~~-10, -10~~

$v_1 = 100$, prob $= 0{,}5$

		Player 2		
		$b_2 = 40$	~~$b_2 = 60$~~	~~$b_2 = 80$~~
	$b_1 = 40$	30, 10	~~0, 0~~	~~0, -20~~
Player 1.100	$b_1 = 60$	40, 0	~~20, 0~~	~~0, -20~~
	$b_1 = 80$	20, 0	~~20, 0~~	~~10, -10~~

Eliminate weakly dominated strategies for 2.

First price auction

- Eliminate weakly dominated strategies for Type $v_1 = 60$: $b_1 = 60$ and $b_1 = 80$ are weakly dominated by $b_1 = 40$.
- Eliminate weakly dominated strategies for Type $v_1 = 100$: $b_1 = 40$ is weakly dominated by $b_1 = 60$.
- Eliminate weakly dominated strategies for Player 2: $b_2 = 60$ and $b_2 = 80$ are weakly dominated by $b_2 = 40$.

First price auction

After the first round of elimination of weakly dominated strategies we get:

$v_1 = 60$, prob $= 0.5$

		Player 2
		$b_2 = 40$
Player 1.60	$b_1 = 40$	10, 10

$v_1 = 100$, prob $= 0.5$

		Player 2
		$b_2 = 40$
Player 1.100	$b_1 = 60$	40, 0
	~~$b_1 = 80$~~	~~20, 0~~

- Now $b_1 = 80$ is weakly dominated for 1.100.
- We are left with one action for each type, and that will be the Bayesian equilibrium: ((40, 60), 40).
- Equilibrium Payoffs: Player 1 obtains $0.5 \times 40 + 0.5 \times 10 = 25$, Player 2 obtains $0.5 \times 10 + 0.5 \times 0 = 5$.
- The auctioneer’s expected revenues are $0.5 \times 40 + 0.5 \times 60 = 50$.

Second price auction

- The player placing the highest bid gets the item. She pays the second highest bid.
- Players: buyers 1 and 2.
- Possible actions : $b_i \in [0, \infty)$, $i = 1, 2.$
- Types: each player has a valuation v_i, that may be their private information.
- Player i‘s utility:

$$u_i(b_1, b_2; v_1, v_2) = \begin{matrix} v_i - b_j & \text{if } b_i > b_j \\ \dfrac{v_i - b_j}{2} & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{matrix}$$

Second price auction

$v_1 = 60$, prob $= 0.5$

		Player 2		
		$b_2 = 40$	$b_2 = 60$	$b_2 = 80$
	~~$b_1 = 40$~~	~~10, 10~~	~~0, 20~~	~~0, 20~~
Player 1.60	$b_1 = 60$	20, 0	0, 0	0, 0
	~~$b_1 = 80$~~	~~20, 0~~	~~0, 0~~	~~-10, -10~~

$v_1 = 100$, prob $= 0.5$

		Player 2		
		$b_2 = 40$	$b_2 = 60$	$b_2 = 80$
	~~$b_1 = 40$~~	~~30, 10~~	~~0, 20~~	~~0, 20~~
Player 1.100	~~$b_1 = 60$~~	~~60, 0~~	~~20, 0~~	~~0, 0~~
	$b_1 = 80$	60, 0	40, 0	10, -10

Eliminate weakly dominated strategies for 1.60 and 1.100.

Second price auction

$v_1 = 60$, prob $= 0.5$

Jugador 2

		~~$b_2 = 40$~~	$b_2 = 60$	~~$b_2 = 80$~~
	$b_1 = 40$	~~10, 10~~	0, 20	~~0, 20~~
Jugador 1.60	$b_1 = 60$	~~20, 0~~	0, 0	~~0, 0~~
	$b_1 = 80$	~~20, 0~~	0, 0	~~-10, -10~~

$v_1 = 100$, prob $= 0.5$

Jugador 2

		~~$b_2 = 40$~~	$b_2 = 60$	~~$b_2 = 80$~~
	$b_1 = 40$	~~30, 10~~	0, 20	~~0, 20~~
Jugador 1.100	$b_1 = 60$	~~60, 0~~	20, 0	~~0, 0~~
	$b_1 = 80$	~~60, 0~~	40, 0	~~10, -10~~

Eliminate weakly dominated strategies for 2.

Second price auction

- Eliminate weakly dominated strategies for Type $v_1 = 60$: $b_1 = 40$ and $b_1 = 80$ are weakly dominated for $b_1 = 60$.
- Eliminate weakly dominated strategies for Type $v_1 = 100$: $b_1 = 40$ and $b_1 = 60$ are weakly dominated for $b_1 = 80$.
- Eliminate weakly dominated strategies for Player 2: $b_2 = 40$ and $b_2 = 80$ are weakly dominated for $b_2 = 60$.
- After the elimination we are left with the Bayesian equilibrium: ((60, 80), 60).
- The equilibrium expected payoffs are:
  - $0.5 \times 0 + 0.5 \times 40 = 20$ for Player 1,
  - $0.5 \times 0 + 0.5 \times 0 = 0$ for Player 2.
- The auctioneer’s expected revenue is $0.5 \times 60 + 0.5 \times 60 = 60$ (recall that he gets the second price).

Cournot: private info. over costs

- Two firms compete on quantity in a market with demand $p = A - Q$.
- Marginal costs for Firm 1 are either c_A or c_B with probabilities 1/3, 2/3 and with $c_A > c_B$.
- Marginal cost for Firm 2 are c_2 .
- Each firm knows its own costs.
- Firm 1 knows the costs for Firm 2.
- Firm 2 does not know the costs for Firm 1, but knows that they are either c_A or c_B with the above probabilities.

Cournot: private info. over costs

- This is the problem for Firm 1:
  - If costs are c_A:

    $$\max_{q_A} (A - q_A - q_2)q_A - c_A q_A,$$

    $$q_A = \frac{A - q_2 - c_A}{2}.$$

  - If costs are c_B:

    $$\max_{q_B} (A - q_B - q_2)q_B - c_B q_B,$$

    $$q_B = \frac{A - q_2 - c_B}{2}.$$

- Firm 2:

  $$\max_{q_2} \frac{1}{3}(A - q_A - q_2 - c_2)q_2 + \frac{2}{3}(A - q_B - q_2 - c_2)q_2,$$

  $$q_2 = \frac{1}{3}\frac{A - q_A - c_2}{2} + \frac{2}{3}\frac{A - q_B - c_2}{2}.$$

- Using the three reaction functions we will find the equilibrium.

Cournot: private info. over the demand

- Two firms compete on quantity in a market with demand $p = A - Q$ where A can take values A_A or A_B, with probabilities ½, ½, and $A_A > A_B$.
- Marginal costs for Firm i are c_i.
- Firm 1 knows with certainty the value for the demand.
- Firm 2 does not know the demand, but knows that it will be either A_A or A_B with the above probabilities.

Cournot: private info. over the demand

- This is the problem for Firm 1:
  - If demand is A_A:

$$\max_{q_A}(A_A - q_A - q_2)q_A - c_1 q_A,$$

$$q_A = \frac{A_A - q_2 - c_1}{2}.$$

  - If demand is A_B:

$$\max_{q_B}(A_B - q_B - q_2)q_B - c_1 q_B,$$

$$q_B = \frac{A_B - q_2 - c_1}{2}.$$

- Firm 2:

$$\max_{q_2}\frac{1}{2}(A_A - q_A - q_2 - c_2)q_2 + \frac{1}{2}(A_B - q_B - q_2 - c_2)q_2,$$

$$q_2 = \frac{1}{2}\frac{A_A - q_A - c_2}{2} + \frac{1}{2}\frac{A_B - q_B - c_2}{2}.$$

- Using the three reaction functions we will find the equilibrium.

Contributions to a public good

- Two players must decide simultaneously whether to contribute towards the provision of a public good.
- Their actions are “contribute” and “do not contribute”.
- Each player gets a utility of 3 if they both contribute, of 1 only one decides to contribute and 0 no one contributes.
- For Player 1, the cost of contributing is $c_1 \in (1,3)$.
- For Player 2, the cost of contributing is either $c_A \in (0,1)$ or $c_B \in (1,3)$, with probabilities 2/5, 3/5 respectively.
- Each player knows his own costs.
- Player 2 knows the costs for Payer 1.
- Player 1 does not know the costs for Player 2, but knows that they must be either c_A or c_B with the above probabilities.

Contributions to a public good

- Payoffs are:

$t_2 = c_A$, Prob 2/5

Player 2

		Contribute	Do not contribute
Player 1	Contribute	$3 - c_1, 3 - c_A$	$1 - c_1, 1$
	Do not contribute	$1, 1 - c_A$	$0, 0$

$t_2 = c_{\mathrm{B}}$, Prob 3/5

Player 2

		Contribute	Do not contribute
Player 1	Contribute	$3 - c_1, 3 - c_B$	$1 - c_1, 1$
	Do not contribute	$1, 1 - c_B$	$0, 0$

Contributions to a public good

- For example, let $c_1 = 1.5$, $c_A = 0.2$ and $c_B = 1.8$:

$c_A = 0.2$, Prob 2/5

Player 2

		Contribute	Do not contribute
Player 1	Contribute	$1.5, 2.8$	$-0.5, 1$
	Do not contribute	$1, 0.8$	$0, 0$

$c_{\mathrm{B}} = 1.8$, Prob 3/5

Player 2

		Contribute	Do not contribute
Player 1	Contribute	$1.5, 1.2$	$-0.5, 1$
	Do not contribute	$1, -0.8$	$0, 0$

Strategies after eliminating “Do not contribute” for $2.c_A$:
1: (C, N),
2: (CC, CN).

Let us build the normal form.

Contributions to a public good

Player 2

Player 1		CC	CN
	C	1.5, 1.84	0.3, 1.72
	N	1, −0.16	0.4, 0.32

- There are two EN in pure strategies : (C, CC) and (N, CN), that will also be BNE in pure strategies
- There will also be a third equilibrium in mixed strategies.