Repeated games

2. Infinite repetition

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Infinite repetition

- A stage game is played at periods 1, 2, 3,… with no end.
- At period t players observe the results of all previous periods, from 1 to $t-1$.
- Each player discounts future payoffs using a discount factor δ, $0 < \delta < 1$.
- The extensive form is infinite and has no final nodes; thus, we have to adapt the definition of a strategy and the payoffs assignment.
- A strategy is now a rule to assign actions to the static game in each stage of a subgame depending on the history leading to that subgame.
- Instead of assigning payoffs to final nodes, we will assign payoffs to every possible strategy as the discounted sum of payoffs obtained in each stage.
- Example: the presented discounted value of obtaining Π in each period is:

$$\Pi + \delta\Pi + \delta^2\Pi + \delta^3\Pi + \cdots = \Pi \sum_{t=0}^{\infty} \delta^t = \frac{1}{1-\delta}\Pi.$$

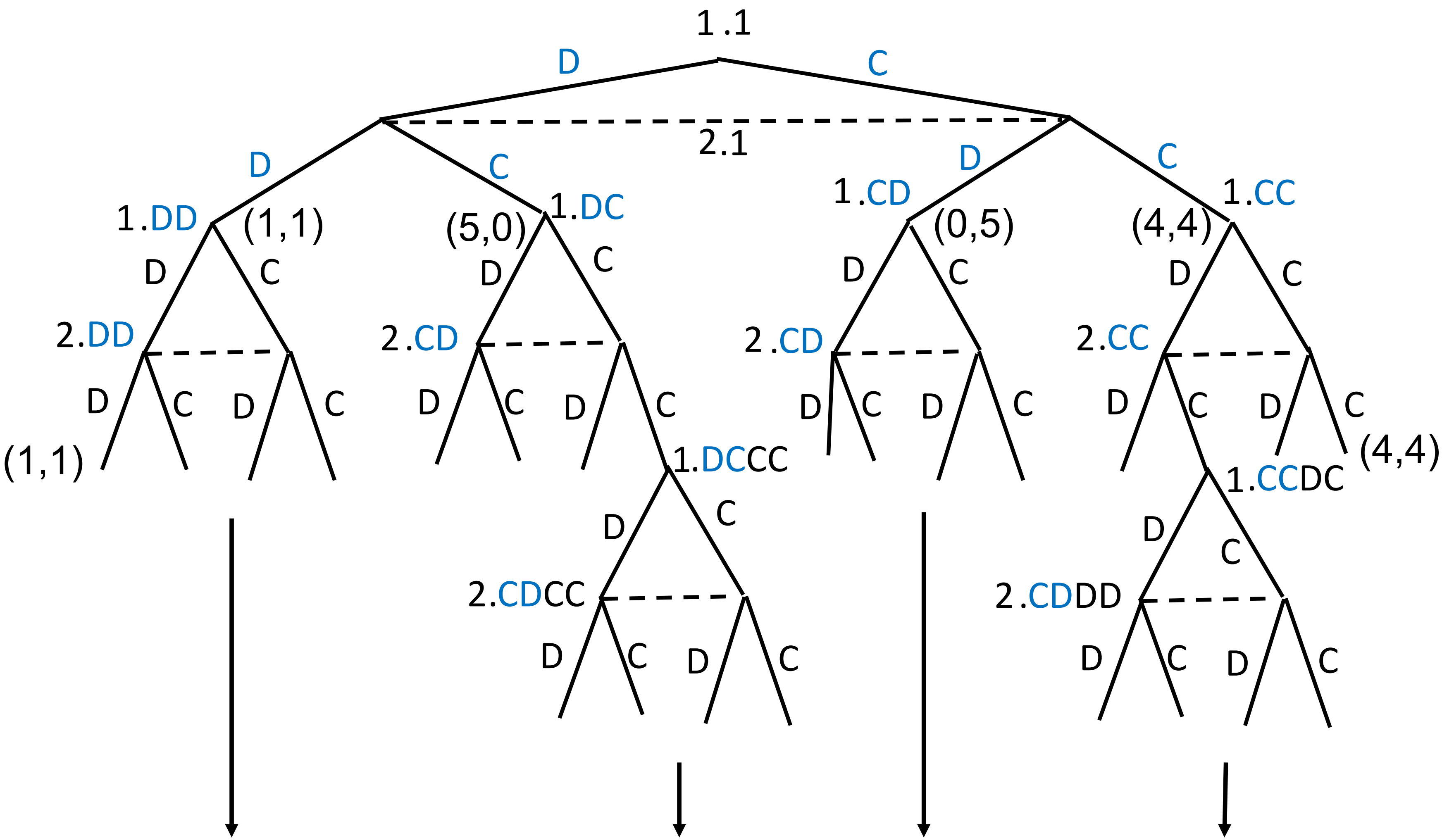
The prisoners' dilemma

- As in the case of finite games , the number of equilibria is potentially huge.
- Rather than finding all of them, we will look for interesting equilibria, typically equilibria that sustain some payoffs.
- E.g., consider the prisoners' dilemma repeated infinitely many times: can we sustain cooperation?

Player 2

Player 1		D	C
	D	1 , 1	5 , 0
	C	0 , 5	4 , 4

Extensive form



Infinitely many repetitions

Trigger strategy

- There are infinitely many subgames.
- Each subgame is identical to the whole game.
- We are interested in sustaining cooperation.
- We will use trigger strategies:
  - Cooperate if cooperation was played at all previous stages (prize).
  - After a deviation, play a NE of the stage game forever (punishment).
- More formally, in our example:
  - At $t = 1$: Play (C, C).
  - At $t > 1$: Play (C, C) if (C, C) was played at all $t' < t$.
    Play (D, D) if at some $t' < t$ (C, C) was not played.

Trigger strategy

- Let us check that the trigger strategy is indeed a SPNE.
- Two steps:
  1. Check it is a NE of the repeated game.
  2. Check it is a NE of each subgame.
- We will make use of two facts:
  - Each subgame is identical to the whole game.
  - The trigger strategy separates subgames in two groups:
    1. Subgames after a history of cooperation.
    2. Subgames after a history that includes some non-cooperative play.

Trigger strategy

At $t = 1$: Play (C, C).
At $t > 1$: Play (C, C) if (C, C) was played at all $t' < t$.
Play (D, D) if at some $t' < t$ (C, C) was not played.

- It is a NE of the repeated game:
- If both players follow the trigger strategy, each one gets:

$$4 + 4\delta + 4\delta^2 + 4\delta^3 + \cdots = \frac{4}{1-\delta}$$

- If one deviates only in the first stage, and plays D she will get:

$$5 + \delta + \delta^2 + \delta^3 + \cdots = 5 + \frac{\delta}{1-\delta}$$

- The deviation is not profitable as long as:

$$\frac{4}{1-\delta} \geq 5 + \frac{\delta}{1-\delta} \qquad \text{or} \qquad \delta \geq \frac{1}{4}.$$

- Have we shown that it is a Nash equilibrium if $\delta \geq \frac{1}{4}$?
- No: we have just shown that one among infinitely many possible deviations is not profitable.

Trigger strategy

At $t = 1$: Play (C, C).
At $t > 1$: Play (C, C) if (C, C) was played at all $t' < t$.
Play (D, D) if at some $t' < t$ (C, C) was not played.

- Which other deviations exist?:
  - Deviations in just one period $t > 1$.
  - Deviations in more than one period.
- If a player deviates to D in just one period at $t > 1$:

$$4 + 4\delta + \cdots + 4\delta^{t-1} + 5\delta^t + \delta^{t+1} + \delta^{t+2} + \cdots,$$

compared to non deviating:

$$4 + 4\delta + \cdots + 4\delta^{t-1} + 4\delta^t + 4\delta^{t+1} + 4\delta^{t+2} + \cdots,$$

If we eliminate the firs t terms $(4 + 4\delta + \cdots + 4\delta^{t-1})$, that are the same in both cases, and divide by δ^t, we are in the same case as when we considered deviations at $t = 1$.

- If a player deviates to D at $t = 1$ and deviates again at t, t', …:

$$5 + \delta + \delta^2 + \cdots + \delta^{t-1} + 0\delta^t + \delta^{t+1} + \cdots + \delta^{t'-1} + 0\delta^{t'} + \delta^{t'+1} + \cdots.$$

We observe that the payoff is smaller than deviating only at $t = 1$:

$$5 + \delta + \delta^2 + \cdots + \delta^{t-1} + \delta^t + \delta^{t+1} + \cdots + \delta^{t'-1} + \delta^{t'} + \delta^{t'+1} + \cdots.$$

- Thus, the best deviation is to deviate in just one period. Now we conclude that the trigger strategy is a NE of the whole game if $\delta \geq \frac{1}{4}$.

Trigger strategy

At $t = 1$: Play (C, C).
At $t > 1$: Play (C, C) if (C, C) was played at all $t' < t$.
Play (D, D) if at some $t' < t$ (C, C) was not played.

- Let us now show that the trigger strategy is a NE in each subgame that is not the whole game.
- Recall that all subgames are equal (the infinite repetition of the prisoners' dilemma) and that the trigger strategy separates subgames in two groups at the time to prescribe the actions to follow:

1) Subgames after a history of cooperation.
2) Subgames after a history that includes at least one period in which there has been no full cooperation.

- Subgames at t after (C, C) has been played for all $t' < t$:
  - In these subgames, the trigger strategy prescribes the same actions as at $t = 1$ and the subgame is the same as the whole game.
  - Thus, the trigger strategy is a NE in those subgames under the same conditions as in the whole game.
- Subgames at t if for some $t' < t$ the actions (C, C) where not played.
  - If players do not deviate, each one will get:

    $\delta^t + \delta^{t+1} + \cdots + \delta^{t'-1} + \delta^{t'} + \delta^{t'+1} + \cdots$.
  - If one deviates at t, t', …:

    $0\delta^t + \delta^{t+1} + \cdots + \delta^{t'-1} + 0\delta^{t'} + \delta^{t'+1} + \cdots$.
  - Thus, the trigger strategy is a NE in those subgames for any value of δ.
- We conclude that the trigger strategy is a SPNE of the game.

Trigger strategy

- There are many other payoffs, besides (1, 1) and (4, 4), that can be sustained in a SPNE. Let us see how to sustain another one.
- Consider the following trigger strategy:
  - At $t = \text{odd}$ play (C, C).
  - At $t = \text{even}$ play (D, C).
  - Keep playing like that if no one deviates.
  - If at some point some one deviates, play (D, D) forever after.
- If they both follow the strategy, Player 1 gets:

$$4 + \delta 5 + \delta^2 4 + \delta^3 5 + \cdots = \frac{4}{1-\delta^2} + \frac{5\delta}{1-\delta^2}.$$

- Player 2 gets:

$$4 + \delta 0 + \delta^2 4 + \delta^3 0 + \cdots = \frac{4}{1-\delta^2}.$$

- For high enough values of δ, the strategy is a SPNE.
- In general, we can design the following trigger strategy, and it will be a SPNE for high enough values of δ:
  - Define a way to play that give the payoffs to be sustained (they must be larger or equal to the payoffs in a NE).
  - Keep playing like that if no one deviates.
  - If some one deviates, change to play the NE forever after.

Undefined or uncertain repetition

- Uncertain repetition: the game continues to the next period with probability p.
- Keep the discount rate δ.
- The expected discount payoff of a series of payments is now calculated as:

$$\Pi_1 + p\delta\Pi_2 + (p\delta)^2\Pi_3 + (p\delta)^3\Pi_4 + \cdots = \sum_{t=1}^{\infty}(p\delta)^{t-1}\,\Pi_t.$$

- In particular, if $\Pi_t = \Pi$ for all t:

$$\sum_{t=1}^{\infty}(p\delta)^{t-1}\,\Pi_t = \Pi\sum_{t=0}^{\infty}(p\delta)^t = \frac{1}{1-p\delta}\Pi.$$

- We observe that the model is equivalent to the one already studied with a discount rate $\delta' = p\delta$.

Application to Cournot oligopoly

- Firms interact infinitely many times (or many times with not a specified final):
  - They can learn no coordinate their strategies.
  - They can threat with some periods of punishment (low profits) in case of a deviation.
- Implications:
  - If firms are patient enough, they can sustain monopoly prices in each period.
  - The larger the number of firms, the more difficult it is to collude.

Application to Cournot oligopoly

- Recall the Cournot game seen in static games (2 firms, linear demand and equal, constant marginal costs:
  $\Pi_i = q_i(a-(q_1+q_2)) - cq_i$.
- The Nash-Cournot equilibrium is: $\left(q_1^C, q_2^C\right) = \left(\frac{a-c}{3}, \frac{a-c}{3}\right)$.
- Equilibrium profits are $\Pi_i^C = \Pi_i^C = \frac{1}{9}(a-c)^2$.
- Recall also that the monopoly quantity maximizes profits and is $q^M = \frac{a-c}{2}$, with profits $\Pi^M = \frac{1}{4}(a-c)^2$.
- Thus, if each produces $q_i^M = \frac{q^M}{2} = \frac{a-c}{4}$, each one will enjoy profits equal to $\Pi_i^M = \frac{\Pi^M}{2} = \frac{1}{8}(a-c)^2$, higher than profits in the Nash-Cournot equilibrium.

Application to Cournot oligopoly

- Trigger strategy:
  - Start cooperating and keep cooperating as long as no one deviates.
  - If someone deviates, stop cooperating.
- Formally:
  - At $t = 1$: Play $(q_1^M, q_2^M) = \left(\frac{a-c}{4}, \frac{a-c}{4}\right)$.
  - At $t > 1$: Play (q_1^M, q_2^M) if (q_1^M, q_2^M) was played at all $t' < t$.

    Play $(q_1^C, q_2^C) = \left(\frac{a-c}{3}, \frac{a-c}{3}\right)$ if at some period $t' < t$ (q_1^M, q_2^M) was not played.
- Differences with the repeated prisoners' dilemma:
  - There are infinitely many strategies in the stage game, not just two.
  - The best deviation in the stage game against q_j^M is not to play q_i^C, but the quantity indicated by the reaction function. Denote this best deviation by q_i^D:

$$q_i^D = \frac{a-c-q_j^M}{2} = \frac{a-c-\frac{a-c}{4}}{2} = \frac{3}{8}(a-c).$$

  - Profits for Firm i after $\left(q_i^D, q_j^M\right)$ are:

$$\Pi_i^D\left(q_i^D, q_j^M\right) = \left(a - \frac{3}{8}(a-c) - \frac{a-c}{4} - c\right)\frac{3}{8}(a-c) = \frac{9}{64}(a-c)^2.$$

Application to Cournot oligopoly

- Let us check that the trigger strategy is a SPNE:
- If both firms follow the trigger strategy, each one has:

$$\Pi_i^M + \delta\Pi_i^M + \delta^2\Pi_i^M + \cdots = \Pi_i^M \frac{1}{1-\delta} = \frac{1}{8}(a-c)^2 \frac{1}{1-\delta}.$$

- If Firm i deviates only during the first period it gets:

$$\Pi_i^D + \delta\Pi_i^C + \delta^2\Pi_i^C + \cdots = \frac{9}{64}(a-c)^2 + \frac{1}{9}(a-c)^2 \frac{\delta}{1-\delta}.$$

- The deviation is not profitable if:

$$\frac{1}{4}(a-c)^2 \frac{1}{1-\delta} \geq \frac{9}{64}(a-c)^2 + \frac{1}{9}(a-c)^2 \frac{\delta}{1-\delta}.$$

- Solving for δ we obtain that $\delta \geq \frac{9}{17}$ is the condition for no deviations by either firm.
- Repeating the same arguments seen in the repeated prisoners' dilemma, we can show that this is the best deviation and, thus, that the trigger strategy is a NE.
- Also repeating those arguments, we can show that it is also a NE in all subgames and, then, a SPNE.

More results with games repeated infinitely many times

- Consider again the infinitely repeated prisoners’ dilemma.
- The trigger strategy is too severe and punishes both players for too long. We can design a strategy that limits the punishment to a small number of periods and that punishes only the deviator.
- The analysis can be extended to more players.
- The analysis can be extended also to the case of casual encounters between two random players within a society:
  - E.g.: in a society with 100 individuals, each day two players chosen at random play the prisoners’ dilemma. All observe their behavior.
  - Rules:
    - Each player is born with the “cooperator” label. The label is visible to all.
    - A player keeps the label if she cooperate with another “cooperator” and if does not cooperate against someone with the label “no cooperator”.
    - If a player does not follow the above strategy, her label changes to “no cooperator”.
  - With those rules, cooperate with a “cooperator” and no cooperate with a “no cooperator” is a SPNE.

Back to games repeated finitely many times

- Consider again the prisoners' dilemma repeated finitely many times.
- According to our analysis, the only el SPNE consists on repeating the NEa in each stage in all subgames.
- However, if the game is repeated, say 200 times, it is hard to think that we cannot expect cooperation during many periods (all except a few at the end).
- Small deviations from the rationality assumption allows that. These are just three examples:
  - One of the players may believe that the number of repetitions is different than the actual number y, and the other player may take that into account.
  - There may be mistakes when keeping track of the periods already played.
  - With some probability, one player may be irrational and always cooperate.