Repeated games

1: Finite repetition

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Finitely repeated games

- A finitely repeated game is a dynamic game in which a simultaneous game (the stage game) is played finitely many times, and the result of each stage is observed before the next one is played.

- Example: Play the prisoners' dilemma several times. The stage game is the simultaneous prisoners' dilemma game.

Results

- If the stage game (the simultaneous game) has only one NE the repeated game has only one SPNE: In the SPNE players' play the strategies in the NE in each stage.
- If the stage game has 2 or more NE, one can find a SPNE where, at some stage, players play a strategy that is not part of a NE of the stage game.

The prisoners' dilemma repeated twice

- Two players play the same simultaneous game twice, at $t = 1$ and at $t = 2$.
- After the first time the game is played (after $t = 1$) the result is observed before playing the second time.
- The payoff in the repeated game is the sum of the payoffs in each stage ($t = 1, t = 2$)
- Which is the SPNE?

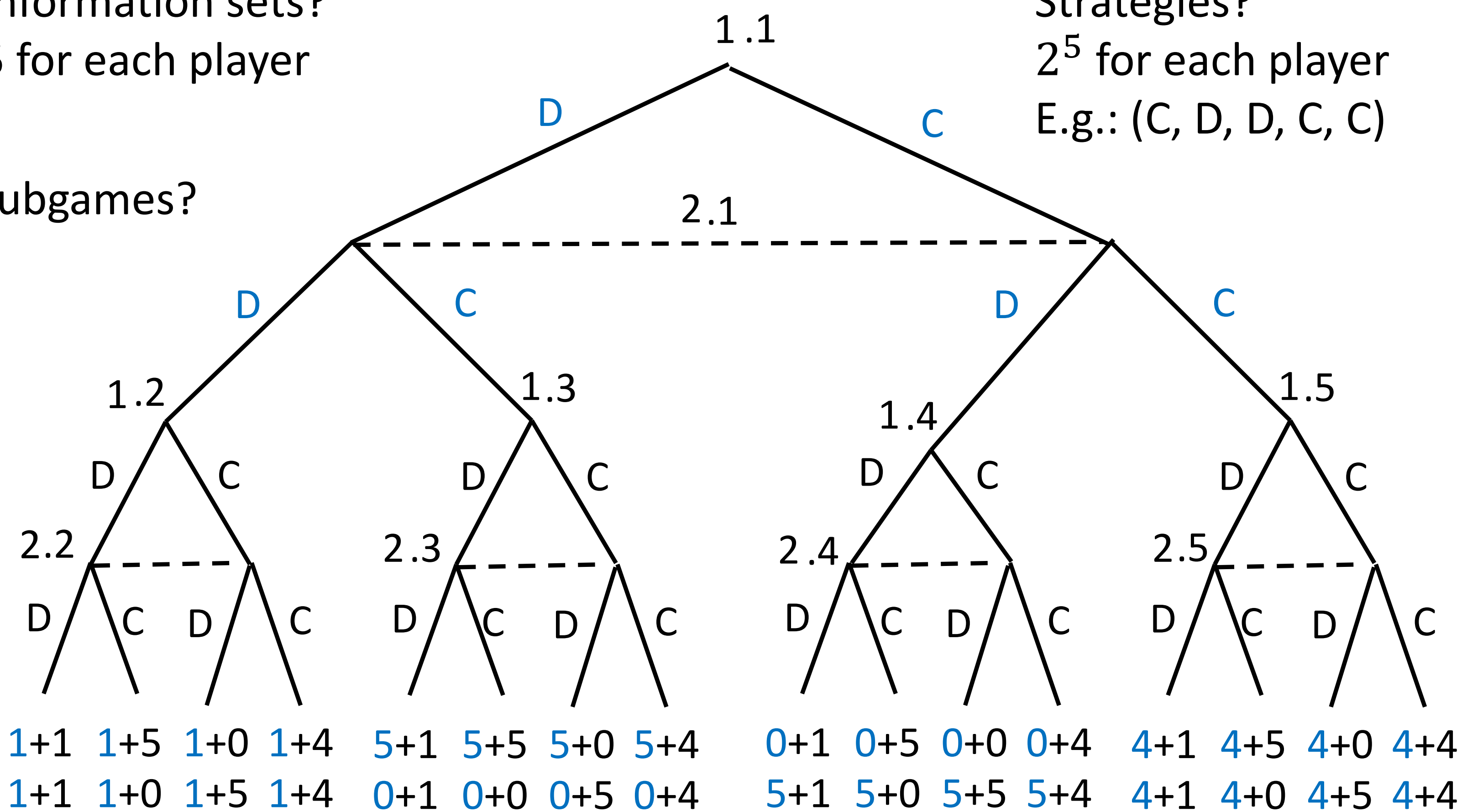
		Player 2	
		D	C
Player 1	D	1 , 1	5 , 0
	C	0 , 5	4 , 4

The prisoners’ dilemma repeated twice

Information sets?
5 for each player

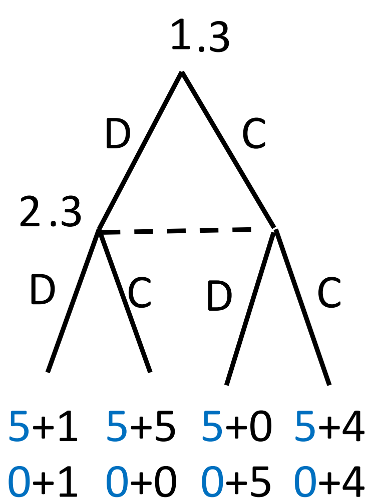
Strategies?
2^5 for each player
E.g.: (C, D, D, C, C)

Subgames?
5



The prisoners’ dilemma repeated twice

Let’s find the NE in the subgames.
Consider the subgame starting at 1.3.
Other subgames are solved similarly.



Player 2

Player 1		D	C
	D	5+1, 0+1	5+5, 0+0
	C	5+0, 0+5	5+4, 0+4

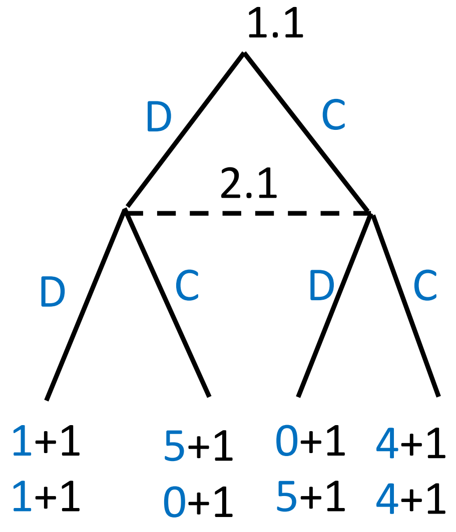
Player 2

Player 1		D	C
	D	1, 1	5, 0
	C	0, 5	4, 4

Adding or not the payoffs of the first stage does not change the game.
In each subgame we have the same prisoners’ dilemma.

The prisoners' dilemma repeated twice

- In each of the four subgames there is only one NE: (D, D).
- Using backward induction, we substitute the payoffs in the equilibria ((1, 1) in all of them) and proceed to solve the first stage.



Player 2

Player 1	D	C
D	1+1, 1+1	5+1, 0+1
C	0+1, 5+1	4+1, 4+1

Player 2

Player 1	D	C
D	1, 1	5, 0
C	0, 5	4, 4

Again, we have a prisoners' dilemma, with NE: (D, D).

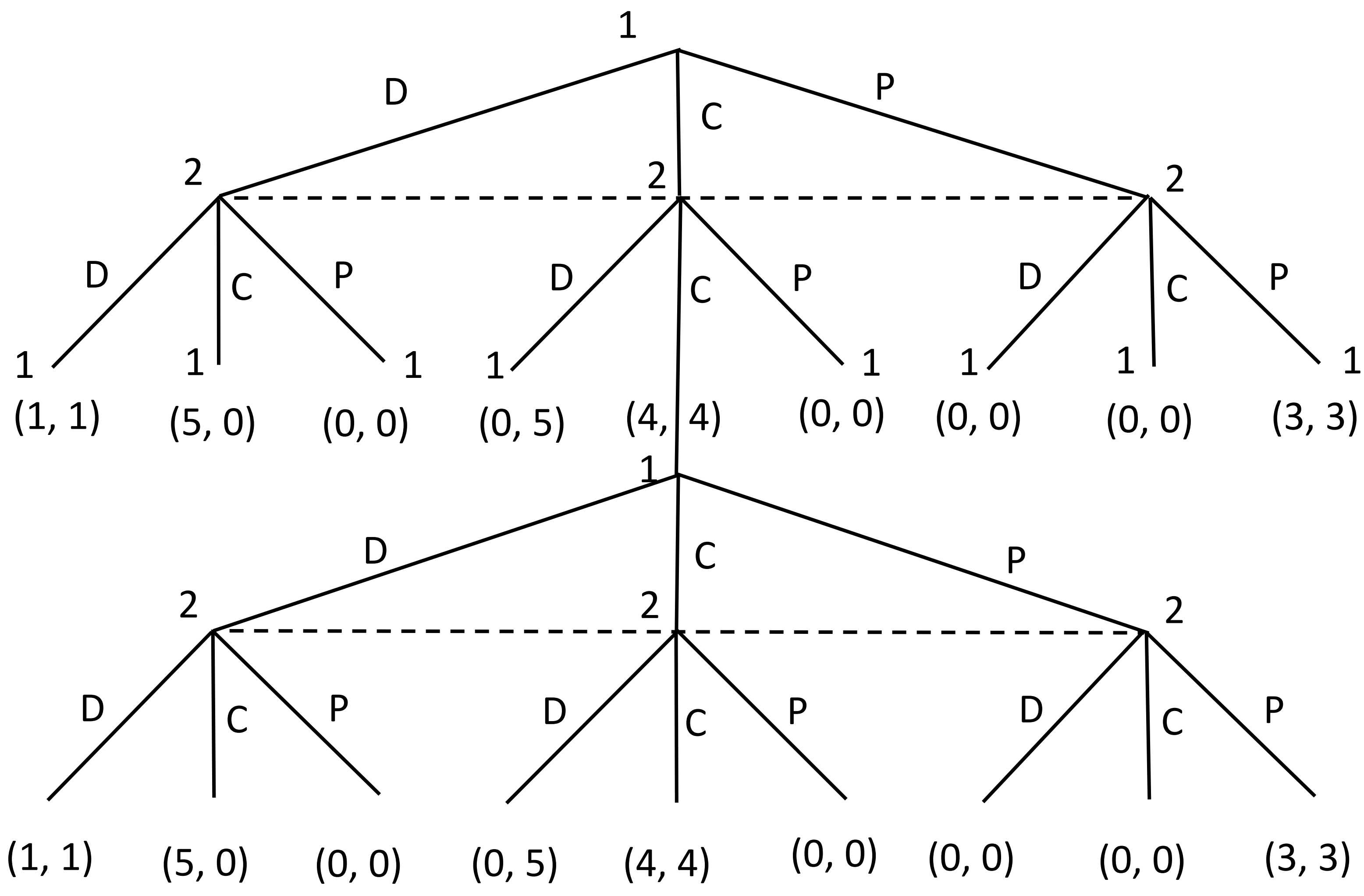
The only SPNE is ((D, D, D, D, D), (D, D, D, D, D)).

The augmented prisoners’ dilemma

		Player 2		
		D	C	P
	D	1 , 1	5 , 0	0 , 0
Player 1	C	0 , 5	4 , 4	0 , 0
	P	0 , 0	0 , 0	3 , 3

- Let’s play twice the game before. It is the prisoners’ dilemma with and added strategy.
- (D, D) is still a Nash equilibrium.
- There is also a new Nash equilibrium: (P, P).
- Could the play of (C, C) be part of a SPNE?
- In the second stage we may not have the same NE in all subgames and, thus, the game in the first stage may not be the same as the stage game.

The augmented prisoners' dilemma



The augmented prisoners' dilemma

- To sustain cooperation (C, C) in the first stage, let's play with the NE in the second one:
  - If there has been cooperation in the first stage, in the second stage play he NE with the best payoffs: (P, P).
  - If there has not been cooperation, in the second stage play the NE with worse payoffs: (D, D).
- Formally:
  - At $t = 1$: (C, C).
  - At $t = 2$: (P, P) if (C, C) was played at $t = 1$,

    (D, D) if (C, C) was not played at $t = 1$.

The augmented prisoners' dilemma

At $t = 1$: (C, C).

At $t = 2$: (P, P) if (C, C) was played at $t = 1$,

(D, D) if (C, C) was not played $t = 1$.

- Let us check if the above strategy is a SPNE:
  - It prescribes a Nash equilibrium in each subgame of the second stage: (P, P) or (D, D).
  - It remains to check whether it is a NE of the whole game.
- If the players follow the strategy, each one gets $4 + 3 = 7$.
- If one player deviates:
  - In the second stage there is no profit in deviating (recall that players play a NE).
  - In the first stage, the best deviation from (C, C) is D. In this first stage, the player that deviates gets 5, but then in the second stage the game reaches a subgame in which the equilibrium is (D, D), with payoff 1.
  - The total for the deviating player is $5 + 1 = 6$, less than the 7 in case of no deviation.
- Thus, the strategy is a SPNE.

The augmented prisoners' dilemma

Recall:

AT $t = 2$: (P, P) if (C, C) was played at $t = 1$,

(D, D) if (C, C) was not player at $t = 1$.

Player 2 (columns), Player 1 (rows)

	D	C	P
D	1 + 1 , 1 + 1	5 + 1 , 0 + 1	0 + 1 , 0 + 1
C	0 + 1 , 5 + 1	4 + 3 , 4 + 3	0 + 1 , 0 + 1
P	0 + 1 , 0 + 1	0 + 1 , 0 + 1	3 + 1, 3 + 1

Player 2 (columns), Player 1 (rows)

	D	C	P
D	$\underline{2}$, $\underline{2}$	6 , 1	1 , 1
C	1 , 6	$\underline{7}$, $\underline{7}$	1 , 1
P	1 , 1	1 , 1	$\underline{4}$, $\underline{4}$

The augmented prisoners' dilemma

- For any combination of NE in the subgames of the second stage, there will be different SPNE.
- How may combinations are there?
- It's 9 subgames, with two NE in pure strategies in each one. Thus, there are $2^9 = 512$ different combinations.
- It will be very complicated to analyze all of them. Instead, we can look for interesting equilibria, like the one we just found.
- Let's see three more examples of SPNE:
  - Play (D, D) always (at $t = 1$ and at $t = 2$ in all subgames).
  - Play (P, P) always.
  - Play (D, D) at $t = 1$ and (P, P) at $t = 2$ in all subgames.
- There are many more. In fact, in these games we typically study the set of payoffs that can be obtained in a SPNE.

The chicken game

	K	S
K	0, 0	5, 1
S	1, 5	4, 4

- The game has 3 NE: $\left\{(\mathrm{S},\mathrm{K}),(\mathrm{K},\mathrm{S}),\left(\frac{1}{2}[\mathrm{K}]+\frac{1}{2}[\mathrm{S}],\frac{1}{2}[\mathrm{K}]+\frac{1}{2}[\mathrm{S}]\right)\right\}$, with respective payoffs: (1, 5), (5, 1), (2,5, 2,5).
- Repeat the game twice. Can we sustain (S, S) in the first stage?
- Define an appropriate strategy:

  At $t=1$: (S, S).

  At $t=2$: $\left(\frac{1}{2}[\mathrm{K}]+\frac{1}{2}[\mathrm{S}],\frac{1}{2}[\mathrm{K}]+\frac{1}{2}[\mathrm{S}]\right)$ if either (S, S) or (K, K) was played at $t=1$,

  (S, K) if (K, S) was played at $t=1$,

  (K, S) if (S, K) was played at $t=1$.
- Observe that this strategy only punishes the unilateral deviator.
- Utility if there are no deviations: $4+2{,}5=6{,}5$.
- Utility for i if he deviates at $t=1$: $5+1=6$.
- The trigger strategy is a SPNE.