Dynamic Games

1. Perfect information

Universidad Carlos III

Dynamic Games

- Games in which a player must make a decision after knowing (part of) the past play, in particular, what some other player did in the past.
- Mathematical model of the game: extensive form.
- Examples: chess, parcheesi, poker.

Dynamic Games

The mathematical model to represent dynamic games is the extensive form, and it must define:

1. the order in which players take their decisions,
2. the information they have at each moment,
3. the set of alternatives available to them,
4. the outcome after each possible path in the game.

Dynamic Games

1. Games with perfect information.
   - All players are perfectly informed of everything that happened up to the moment they play. If there are random moves, all players must have the same information about them.
   - Example: chess, parcheesi.

2. Games with imperfect information.
   - Some player is unaware of the result of some chance move or of the past play of a rival, or if there are random moves, players may have different information about them.
   - Examples: simultaneous games, poker.

Dynamic battle of the sexes

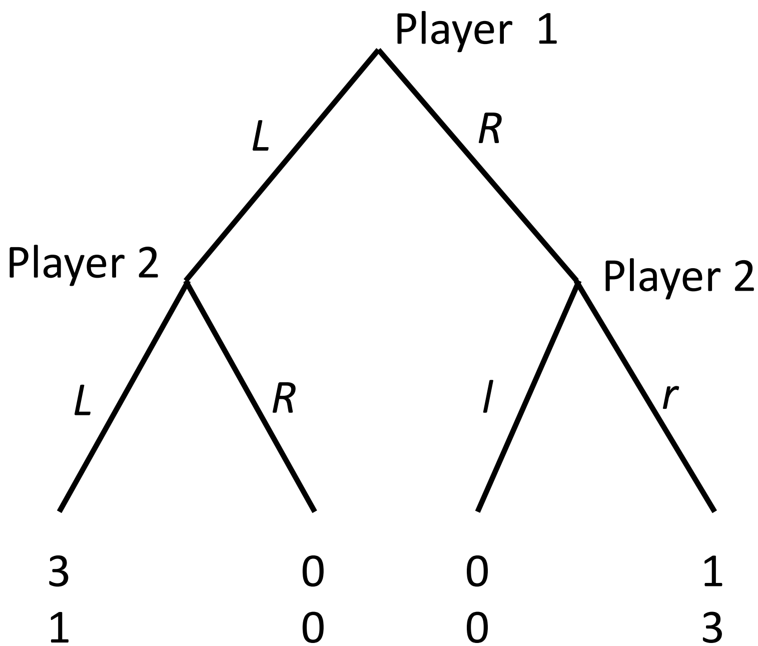
- Recall the battle of the sexes:

Player 2

		L	*R*
Player 1	*L*	<u>3</u>, <u>1</u>	0, 0
	R	0, 0	<u>1</u>, <u>3</u>

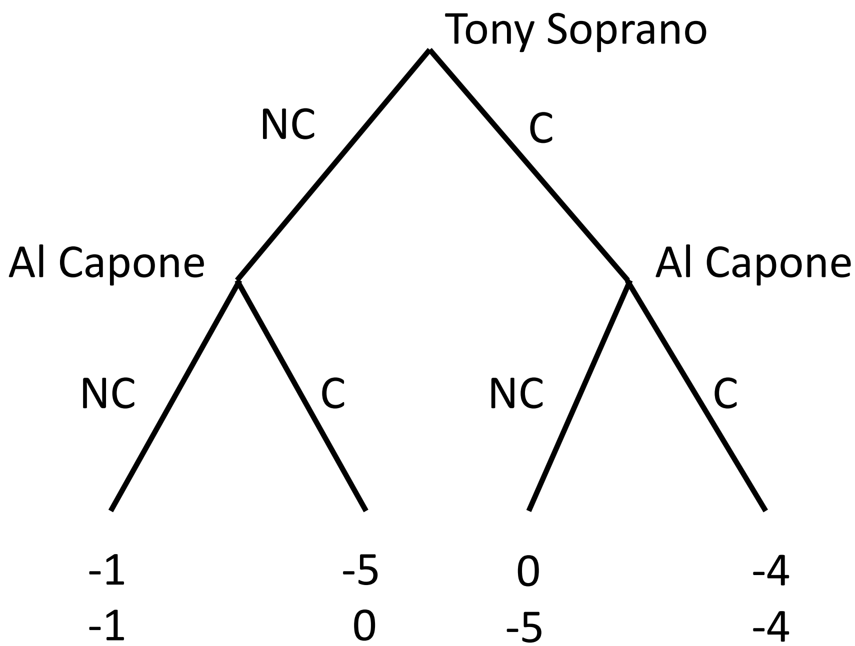
- How can Player 1 change the game in her favor?
  - Send the message: "I am playing *L*".
  - Play "*L*".
- In both cases Player 2 must observe what Player 1 does.
- In the first case, Player 2 must not be able to respond with another message.

Dynamic battle of the sexes



- Which will be the equilibrium in this game?
  - (L, L)?
  - No: better $(L, (L, r))$.

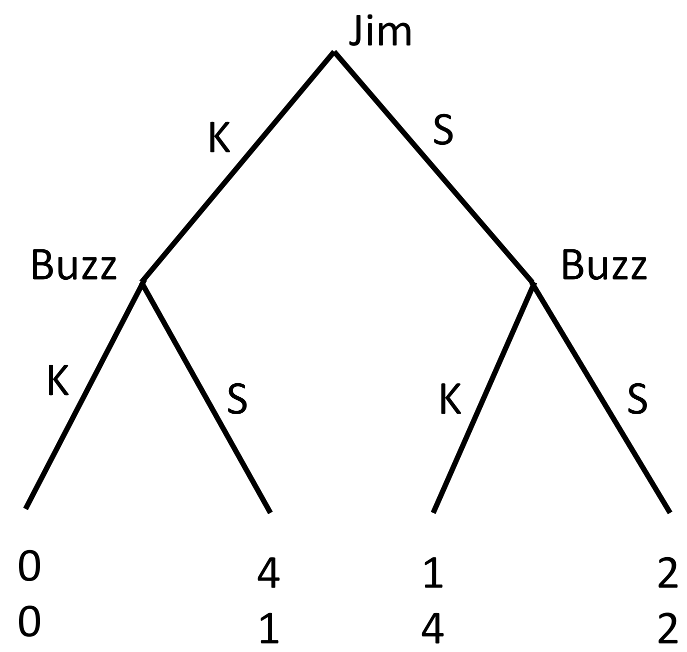
Dynamic prisoners' dilemma



- Which will be the equilibrium in this game?
  - (C, C)?
  - No: better $(C, (C, C))$.
  - We need to specify what will Al Capone do in both contingencies in order to justify why Tony Soprano decides *C*.

Dynamic chicken game

Jim plays first announcing a strategy in a credible way or committing to it.

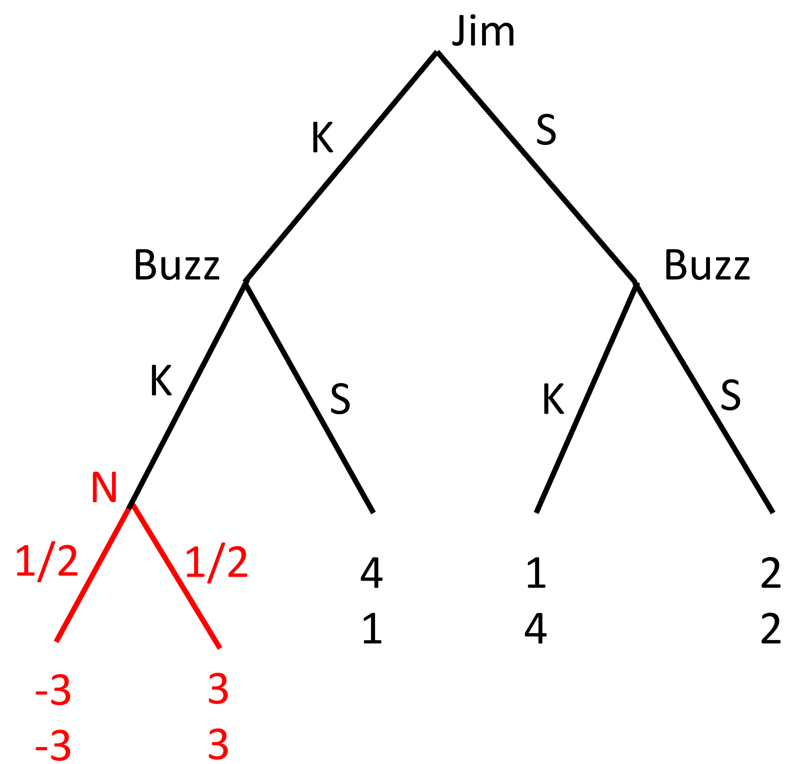


- Which will be the equilibrium in this game?
  - (K, S)?
  - No: better $(K, (S, K))$.

Dynamic chicken game

Let us introduce a nature move.

If both keep going, they die with probability ½.
If they don't die, payoffs are midway between winning and being one of two cowards.
(Think as if we obtained the 0 payoffs as an average of this.)



- Which will be the equilibrium in this game?
  - (K, S)?
  - No: better $(K, (S, K))$.

Extensive form with perfect information

- All players know the past play.
- Elements in the extensive form:
  1) Tree: an initial node from which some edges emerge that arrive to other nodes. From these nodes other edges may emerge and so on. A node from which no edge emerge is called a final node.
  2) Assignment of non final nodes among players: Each non final node must be assigned to only and only one player.
  3) Assignment of actions to each player in each one of his nodes.
  4) Payoff (utilities) assignment: A payoff for each player will be assigned in every final node.
  5) We can incorporate random moves as a new player (nature) whose edges are the events. In this case we need to specify the probabilities of the events.

Extensive form with perfect information

- The first number in each final node is the Payoff for Jim, and the second number, the payoff for Buzz. Thus, if Jim plays *K* and Buzz plays *S*, then payoffs are 0 for Jim and 0 for Buzz.

- Jim has only one node, while Buzz has two.

- The letters in the edges identify players' actions: Jim can choose between actions *K* and *S*, and Buzz can choose between *K* and *R* in his first node, and also in his second node.

From the extensive to the normal form

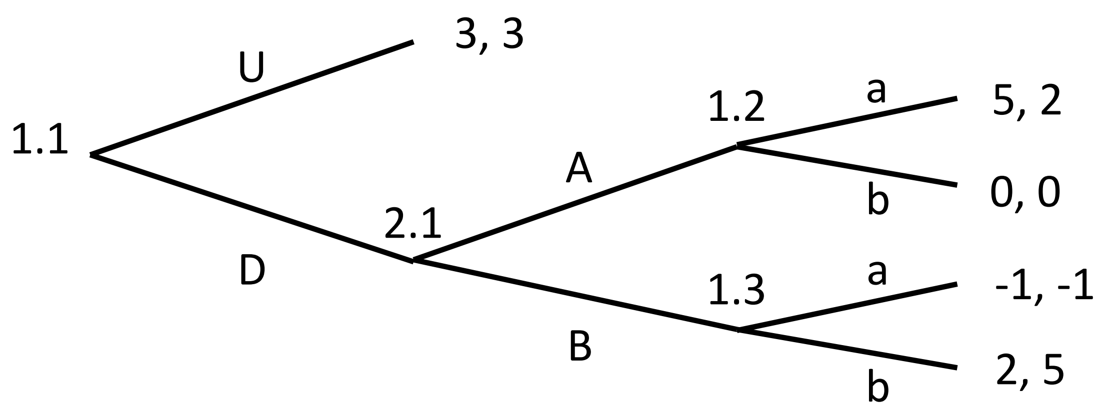
- Recall the dynamic battle of the sexes game.
- Player 1 has two strategies: *L* and *R*. Player 2 has to nodes, in the first one he has two actions (*L* and *R*) and in the second one, another two (*l* and *r*).
- Player 2 has four strategies: (L, l), (L, r), (R, l) and (R, r). The normal form is:

P1 \ P2	(L, l)	(L, r)	(R, l)	(R, r)
L	3, 1	3, 1	0, 0	0, 0
R	0, 0	1, 3	0, 0	1, 3

- Strategy (L, r), for instance, must be read like this: Player 2 chooses *L* in his first node (i.e., if Player 1 has played *L*), and *r* in his second node (if 1 has played *R*).

From the extensive to the normal form

- Let us consider another example.
- The next figure illustrates the extensive form of a perfect information game.
- Observe that Player 1 plays at two different moments (the second time, in one of two possible nodes).



From the extensive to the normal form

Esta es la forma extensiva del ejemplo anterior:

1 \ 2	A	B
(U, a, a)	3, 3	3, 3
(U, a, b)	3, 3	3, 3
(U, b, a)	3, 3	3, 3
(U, b, b)	3, 3	3, 3
(D, a, a)	5, 2	-1, -1
(D, a, b)	5, 2	2, 5
(D, b, a)	0, 0	-1, -1
(D, b, b)	0, 0	2, 5

From the extensive to the normal form

- A strategy in a dynamic game is a contingent plan that specifies what action the player will take in each one of the possible situations (nodes) in which she may move, even in those that come after the initial plan is not followed.

- Observe that the definition of a strategy does not require that all actions specified in it are actually played. Which actions will be played depends on the development of the game.

- To know what a player would hypothetically play in situations that are not reached if the equilibrium is followed allows us to argue why the equilibrium is played.

- In the previous example we can make the following simplification: we can identify strategies (U,a,a), (U,a,b), (U,b,a) and (U,b,b) as just one strategy, that we may call (U). The reason is that those four strategies are indistinguishable as they give the same payoffs for both players.

From the extensive to the normal form

- Normal form: players, strategies and payoffs. We must define those three elements using the extensive form:

  - Set of players. The same.

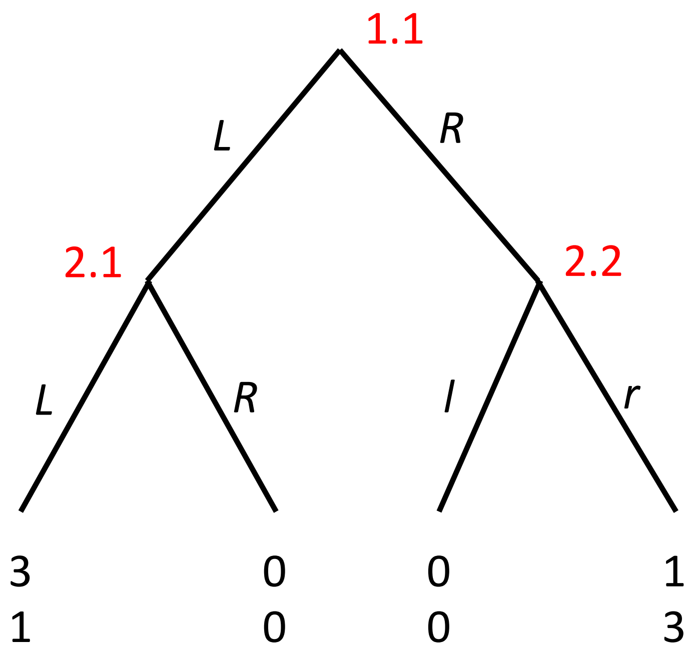
  - Set of strategies. To define he set of strategies of Player i consider the set of nodes that belong to this player. Denote them by $i.1, i.2, \ldots, i.n_i$. Let $A_{i.k}$ be the set of actions that Player i can choose from at $i.k$. A strategy for Player i is then an element of the cartesian product $A_{i.1} \times A_{i.2} \times \cdots \times A_{i.n_i}$.

  - Payoffs. The payoffs that correspond to a certain strategy profile will be obtained from the payoffs of the extensive form that are found after following the actions that define the strategies.

Equilibrium

- Two ways to find the equilibrium:
    1. Backward induction: start by finding the nodes that precede the final nodes and, in each one of them, find the action that maximizes the payoff of the player to whom the node belongs.
    2. Subgame perfect Nash equilibrium (SPNE): The SPNE extends the definition of equilibrium for dynamic games.
- Backward induction: simple, but not always applicable.
- SPNE: more complicated, but always exists.
- We'll see why the notion of NE is not appropriate for dynamic games.

Equilibrium



- Recall the dynamic battle of the sexes.
- Backward induction:
  - at node 2.1 Player 2 will choose *L*,
  - at 2.2 he will choose *r*,
  - anticipating those actions, Player 1 will choose *L* at her only node 1.1.
- Hence: 1 chooses *L* and 2 chooses plan (L, r).
- Backward induction solution is $(L, (L, r))$.

Compare with the set of Nash equilibria in pure strategies:

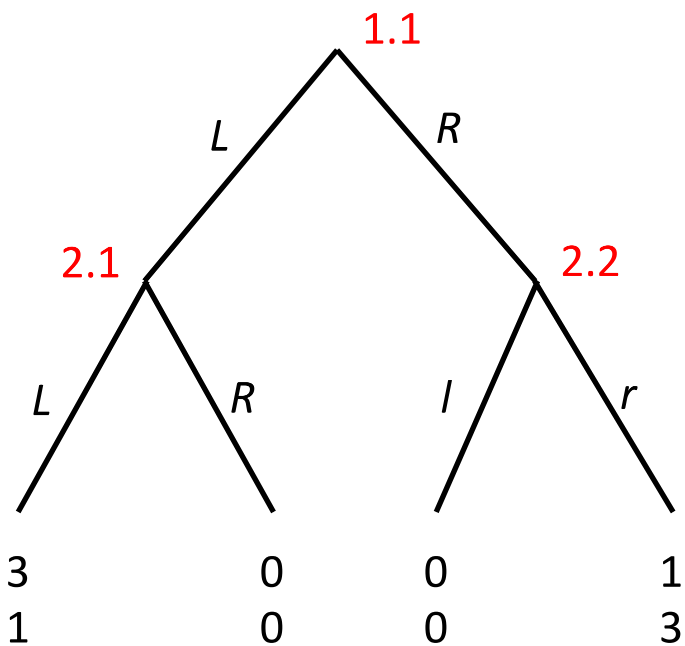
$$NE = \{(L, (L, l)), (L, (L, r)), (R, (R, r))\}.$$

P1 \ P2	(L, l)	(L, r)	(R, l)	(R, r)
L	$\underline{3}$, $\underline{1}$	$\underline{3}$, $\underline{1}$	$\underline{0}$, 0	0, 0
R	0, 0	1, $\underline{3}$	$\underline{0}$, 0	$\underline{1}$, $\underline{3}$

Equilibrium

- Definition of subgame perfect Nash equilibrium (SPNE).
  - Subgame: A subgame in a dynamic game of perfect information consists on a non-final node of the game and all edges and nodes that follow, with the corresponding final payoffs and maintaining the assignment of nodes to players.
  - SPNE: A strategy profile is a SPNE if it is a NE in every subgame.

Equilibrium



Let us go back to the dynamic battle of the sexes to find the SPNE:

- There are 3 subgames, one starting at node 1.1, another one starting at 2.1, and a last one starting at 2.2. (Note that the whole game is also a subgame).
- Two ways to find SPNEa:
  - Check witch NEa satisfy the definition of SPNE.
  - Build directly the SPNEa.
- Recall: $NE = \{(L,(L,l)),(L,(L,r)),(R,(R,r))\}$

Only $(L,(L,r))$ is a SPNE.

Normal form of subgame after 2.1

Player 2

L	*R*
1	0

$NE = \{L\}$

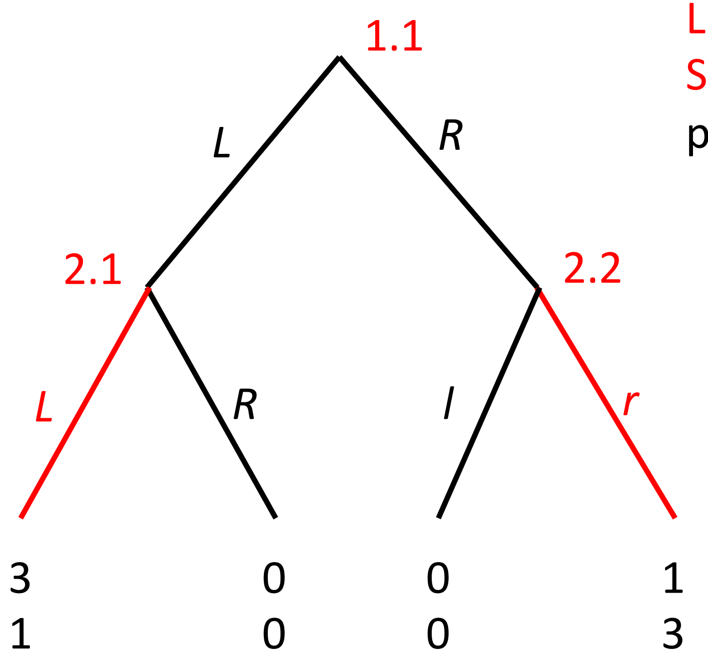
Normal form of subgame after 2.2

Player 2

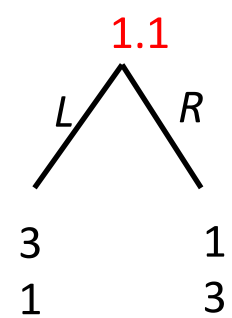
l	*r*
0	3

$NE = \{r\}$

Equilibrium



Let us build the SPNE without knowing the NE
Substitute subgames with the corresponding payoffs in the NE.



Normal form of subgame after 1.1

Player 1

L	R
3	1

$$NE = \{L\}$$

Take all actions in the equilibria of the subgames to get:

$$\text{SPNE} = \{(L, (L, r))\}.$$

Backward induction and SPNE select the same set of equilibria.
This is true for all games of perfect information.

Equilibrium

- Out of the three Nash equilibria in the example, only one is a SPNE. Observe that, if Player 1 chooses *R*, the payoffs are unaffected by Player 2' actions in 2.1: both *L* and *R* are best replies in this case.

- However, if Player 2 must play at 2.1, what matters are the payoffs at that moment, and then only *L* is the best reply.

- This time inconsistency is what the notion of subgame perfect equilibrium tries to avoid.

Equilibrium

- Equilibrium. A strategy profile (one strategy for each player) that satisfies a certain definition (e.g. Backward induction, SPNE). It selects one action for each player at each node.

- Equilibrium path. It is the path (nodes and edges) that is observed if all players play according to their equilibrium strategy. (In the dynamic battle of the sexes, the SPNE path is that Player 1 chooses *L* and 2.1 chooses *L*).

- Equilibrium payoffs. Final payoffs that are obtained if all players follow their equilibrium strategy. (In the dynamic battle of the sexes, backward induction and SPNE give equilibrium payoffs (3,1)).