STATIC GAMES

4. Continuous variable and economic applications

Universidad Carlos III de Madrid

Continuous variable

- In many games, pure strategies that players can choose are not only 2, 3 or any other finite number, sometimes there are infinitely many.
  - Consider two oligopolistic firms must decide how much to sell. The strategy space is the interval $[0, K_i]$, where K_i is the maximum capacity of Firm *i*.
  - A mixed strategy is another example of continuous variable

- We will solve these games by finding the best-reply function (also called best response function or reaction function of each player to the possible actions of his rivals.

- (In general, the best reply is not a function but a correspondence.)

Continuous variable

- Two cases:
  - Differentiable payoff function (e.g., quantity competition problems, voluntary contributions to public goods.)
    - Best response function arises from the first order condition of maximization (if the objective function is concave.)
    - For non-concave functions, we need to consider higher order conditions.
  - Linear or non-differentiable payoff function (e.g.: allocation and distribution problems, price competition.)


Airbus VS Boeing
A330-300 AIRBUS
BOEING


OPEC


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fenosa
IBERDROLA
endesa
edp
e.on

movistar

vodafone

orange™

Cournot model

- Simultaneous competition in quantities: $(q_1, q_2, \dots, q_n)$.
- Homogeneous product: $Q = \sum_{i=1}^{n} q_i$.
- There is a market clearing price: p.

Direct demand: $Q = D(p), D'(p) < 0$

Inverse demand: $p = p(Q)$

Costs: $C_1(q_1), \dots, C_n(q_n)$

Objective: $\max_{q_i \geq 0} \Pi_i = p(Q)q_i - C_i(q_i)$

Cournot model

- A good is produced by two firms: 1 y 2. They produce quantities q_1 and q_2, respectively. Each firm chooses its quantity without knowing the rival's decision.

- Simple example:
  - The market price is $p(Q) = a - Q$, where *a* is a constant and $Q = q_1 + q_2$.
  - The cost for Firm *i* when it produces q_i is $C_i(qi) = ciqi$.
  - Even simpler: $c_1 = c_2 = c$, con $c < a$.

Cournot model

The normal form game is:

- Players: {Firm 1, Firm 2}
- Strategies: $S_1 = \{q_1 \in [0, \infty)\}, S_2 = \{q_2 \in [0, \infty)\}$
- Payoffs:

$$u_1(q_1, q_2) = \Pi_1 = q_1(a - (q_1 + q_2)) - cq_1,$$
$$u_2(q_1, q_2) = \Pi_2 = q_2(a - (q_1 + q_2)) - cq_2.$$

Observe that payoffs are a function of strategies $q_1 \in S_1, q_1 \in S_1$ (a and c are parameters).

Cournot model

The Nash equilibrium is:

- A pair of quantities (q_1^*, q_2^*) such that:

  q_1^* is the best reply by Firm1 against q_2^*,

  q_2^* is the best reply by Firm2 against q_1^*.

- This means that:
  - q_1^* solves $\max_{q_1 \geq 0} q_1(a - (q_1 + q_2^*)) - cq_1$,
  - q_2^* solves $\max_{q_2 \geq 0} q_2(a - (q_1^* + q_2)) - cq_2$.

Obs.: because the strategy is continuous and the payoff function is differentiable, we can find the best reply using calculus techniques.

Cournot model

Finding the NE through the best reply functions:

- From $\max_{q_1 \geq 0} q_1(a - (q_1 + q_2)) - cq_1$

  we get $a - 2q_1 - q_2 - c = 0$ if $q_1 \geq 0$:

$$q_1 = BR_1(q_2) = \max\left\{0, \frac{a - q_2 - c}{2}\right\}.$$

- Analogously for Firm 2:

$$q_2 = BR_2(q_1) = \max\left\{0, \frac{a - q_1 - c}{2}\right\}.$$

Obs.: second order conditions for a maximum are satisfied.

Cournot model

Let us see if the solution corresponds to the case $\frac{a-q_i^*-c}{2} \geq 0$, or $q_i^* \leq a-c$.

The system formed by the best reply functions:

$$q_1 = \frac{a-q_2-c}{2},$$

$$q_2 = \frac{a-q_1-c}{2},$$

has as a result

$$q_1^* = q_2^* = \frac{a-c}{3}.$$

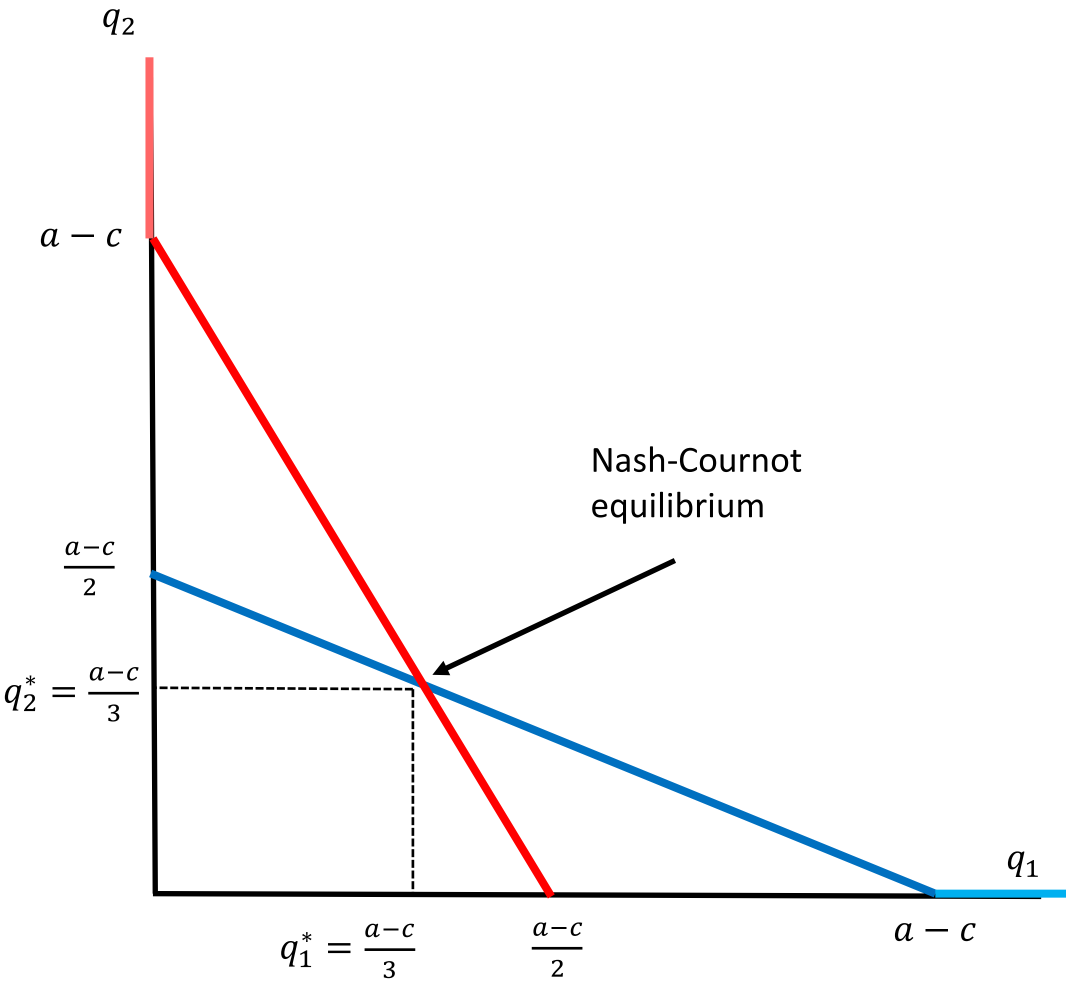
(The solution is positive and smaller than $a-c$, as assumed).

Cournot model

Draw the *BR* functions:

$$q_1 = BR_1(q_2) = \frac{a-q_2-c}{2} \text{ if } q_2 \le a-c,$$
$$= 0 \text{ otherwise.}$$

$$q_2 = BR_2(q_1) = \frac{a-q_1-c}{2} \text{ if } q_1 \le a-c,$$
$$= 0 \text{ otherwise.}$$



Cournot model

- With more tan two firms and constant, identical marginal costs, the best reply functions are of the form:

$$q_i = BR_i(q_2) = \max\left\{0, \frac{a - \sum_{j \neq i} q_j - c}{2}\right\},$$

with them, form a system with *n* equations and *n* unknowns whose solution is

$$q_1^* = q_2^* = \cdots = q_n^* = \frac{a-c}{n+1}.$$

- With two firms and constant, but different marginal costs, the best reply functions are of the form:

$$q_i = MR_i(q_j) = \max\left\{0, \frac{a - q_j - c_i}{2}\right\},$$

with solution $q_i = \frac{a - 2c_i + c_j}{3}$.

Cournot model

1. Price above marginal cost. (The result is not socially efficient.)
   There are incentives to reduce production if competitors produce at marginal cost.

2. Price inferior to monopolistic price. (The result is not efficient if we are restricted to the set of the two firms.)
   Incentives to increase production if competitors produce the monopoly quantity.

3. If the number of firms increases, the price decreases.
   The price goes to the marginal cost as the number of firms goes to infinity.

4. Own quantity decreases with rival's. When this happens in a game, strategic variables are called strategic substitutes.

5. Own quantity decreases with own cost and increases with the rival's.

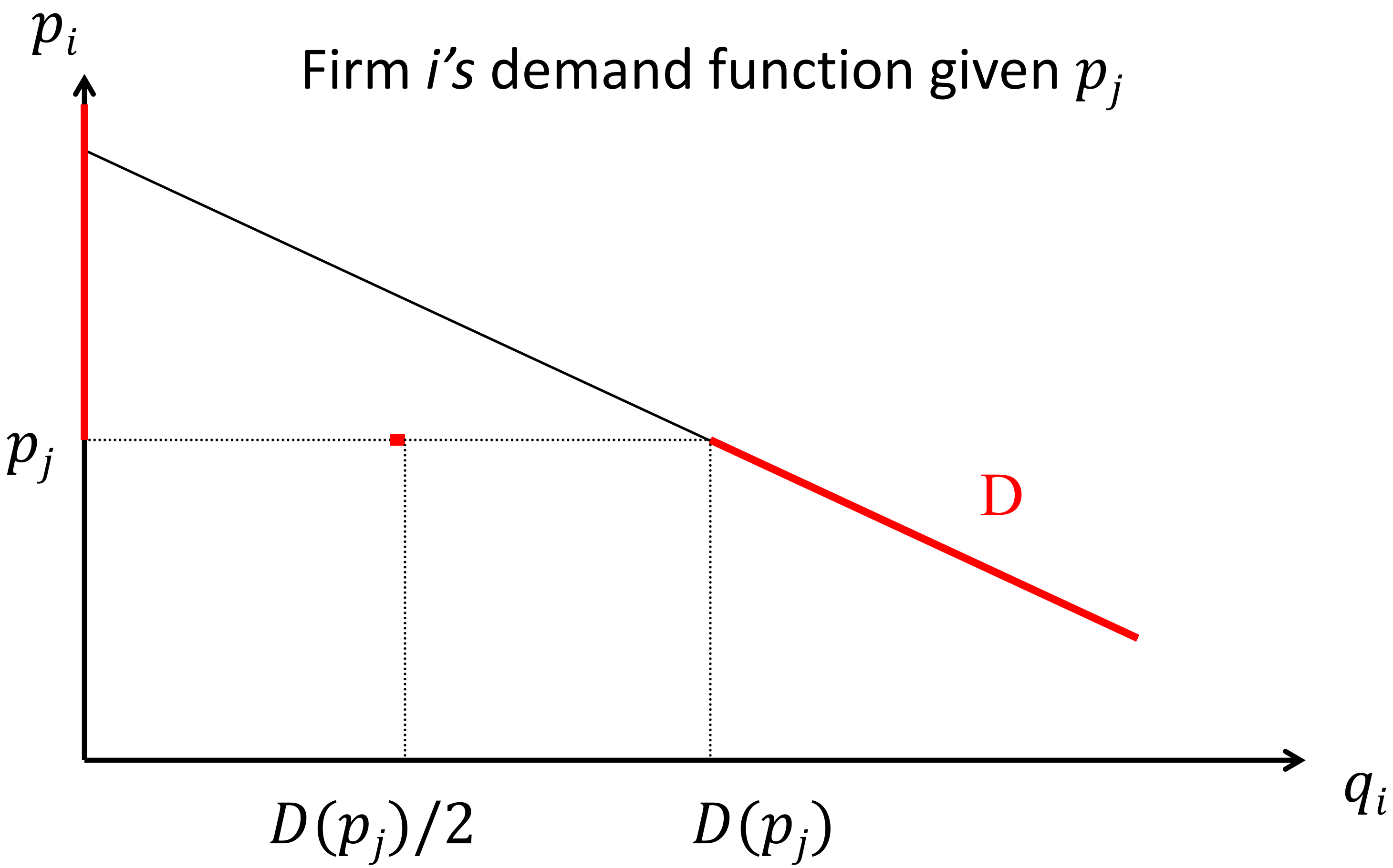
Bertrand model

The model

1. Homogeneous good.
2. Simultaneous competition in prices .
3. Consumers buy from the firm offering the lowest price. If prices are equal, they are indifferent.
4. No capacity restrictions, either firm can produce any quantity at marginal cost *c*.

$$q_i = \begin{cases} D(p_i) & \text{if } p_i < p_j \\ D(p_i)/2 & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

Bertrand model



Note that the demand is discontinuous and therefore we cannot use calculus techniques.

Bertrand model

- The Normal form with demand $p(Q) = a - Q$ and identical marginal costs *c*.
  - Players: {Firm 1, Firm 2}
  - Strategies: $S_1 = \{p_1 \in [0, \infty)\}, S_2 = \{p_2 \in [0, \infty)\}$
  - Payoff functions:

$$u_1(p_1, p_2) = \begin{cases} (p_1 - c)(a - p_1) & \text{if} \quad p_1 < p_2 \\ (p_1 - c)(a - p_1)/2 & \text{if} \quad p_1 = p_2 \\ 0 & \text{if} \quad p_1 > p_2 \end{cases}$$

$$u_2(p_1, p_2) = \begin{cases} (p_2 - c)(a - p_2) & \text{if} \quad p_2 < p_1 \\ (p_2 - c)(a - p_2)/2 & \text{if} \quad p_2 = p_1 \\ 0 & \text{if} \quad p_2 > p_1 \end{cases}$$

Bertrand model

The reaction (best reply) function:

$$p_1(p_2) = \begin{cases} c & \text{if} \quad p_2 \leq c \\ p_2 - \varepsilon & \text{if} \quad p^{monopoly} \geq p_2 > c \\ p^{monopoly} & \text{if} \quad p_2 > p^{monopoly} \end{cases}$$

$$p_2(p_1) = \begin{cases} c & \text{if} \quad p_1 \leq c \\ p_1 - \varepsilon & \text{if} \quad p^{monopoly} \geq p_1 > c \\ p^{monopoly} & \text{if} \quad p_1 > p^{monopoly} \end{cases}$$

Notice that if $p_j \leq c$ the best reply is any price superior or equal to the rival's. For simplicity we consider that in this case the firm chooses $p = c$.

Bertrand model

Let us find the Nash equilibria by trial and error after observing the best reply functions:

- $p_1 > p_2 > c$? No, Firm 1 will deviate to a price between p_2 and *c*.
- $p_1 = p_2 > c$? No, Firm *i* will deviate to a price between p_i and *c*.
- $p_1 > p_2 = c$? No, Firm 2 will deviate to a price between p_2 and p_1.
- $p_1 = p_2 = c$? Yes: in this situation they win 0 each, but any deviation also gives 0 or negative profits.

Bertrand model

- Bertrand’s paradox:
  - Two firms are enough to achieve perfect competition.
- Bertrand’s paradox arises because of the discontinuity of the demand, which implies the discontinuity of profits.
  - The firm selling at the lowest price serves the whole market.
- How to escape the Bertrand’s paradox?
  - Differentiated products.
  - Choose capacity and, then, price.

Price competition with differentiated products

- Two firms produce differentiated products. Both choose prices to maximize profits.
- Linear demands and constant marginal costs.
- Let p_1 be the price of Firm 1 and p_2 the price of Firm 2. Given these prices, Firms' demands are

$$q_1 = a - p_1 + bp_2,$$
$$q_2 = a - p_2 + bp_1.$$

- Both firms have constant marginal costs equal to *c*.

Price competition with differentiated products

- Firm 1 maximizes profits with respect to its strategic variable, its own price:

$$\max_{p_1 \geq 0} (p_1 - c)(a - p_1 + bp_2)$$

- F.O.C.: $a - 2p_1 + bp_2 + c = 0$
- Reaction function: $p_1(p_2) = \frac{a+c+bp_2}{2}$

- We do the same thing for Firm 2.
- The NE is the calculated solving the system formed by the two equations given by the reaction functions taking into account that prices must be higher than *c.*
- See that as rival's price increase, the best response is to also increase the own price: prices are strategic complements.

Voluntary contributions to public goods

- Two consumers, 1 and 2, must decide how much to contribute to a public good. Denote by c_1 and c_2 the respective contributions. Consumer 1 has a net wealth of w_1 whereas consumer 2 has w_2.

- The total quantity of public goods is the sum of the contributions.

- Payoffs:
  - $u_1(c_1, c_2) = (c_1 + c_2)(w_1 - c_1)$,
  - $u_2(c_1, c_2) = (c_1 + c_2)(w_2 - c_2)$.

Voluntary contributions to public goods

- Payoff functions are differentiable:
  - $\max_{0 \le c_1 \le w_1} u_1(c_1, c_2) = (c_1 + c_2)(w_1 - c_1)$
  - FOC give: $c_1 = \frac{w_1 - c_2}{2}$
  - Similarly for Consumer 2: $c_2 = \frac{w_2 - c_1}{2}$.
- Calculate the Nash equilibrium solving the system formed by the best reply functions, the result is:
  - $c_1 = \frac{2w_1 - w_2}{3}$, $c_2 = \frac{2w_2 - w_1}{3}$.
- The solution can be easily shown to be inefficient.