Static games

3. Mixed strategies

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The need of mixed strategies

- We have seen games with no Nash equilibrium (e.g., “Matching pennies”)
- It is easy to see that in this game the best strategy is to be unpredictable and play “heads” and “tails” with probabilities 50:50.
- A player’s mixed strategy is just the choice of a strategy using a probability distribution over her set of strategies.
- From now on we will use the following terms:
  - Pure strategy: each of the original ones.
  - Mixed strategy: each of the probability distributions over the set of a player’s pure strategies.
  - Strategy: each one of the pure and mixed strategies.
- A pure strategy can be seen as a mixed strategy with a degenerated probability distribution.
- The definition of a NE is the same. We just enlarge the set of strategies.

Nadal vs. Federer

- Nadal serves to the left or to the right.
- Federer decides his position for the return, left or right (there is no time to see the ball coming).
- The probability of Nadal winning the point depends of the service and where is Federer positioned:

		Federer	
		Left	Right
Nadal	Left	50 %, 50 %	90 %, 10 %
	Right	70 %, 30 %	50 %, 50 %

- How to play?: Nash equilibrium in mixed strategies.
- First we'll see that they must choose randomly.
- Then we'll see that playing 50:50 is not a NE.
- Finally, we calculate the equilibrium in two different ways.

Nadal vs. Federer

- If Nadal serves always to the same side, Federer will always return in that side
- The same is true if Nadal serves too often to the same side.
- If Federer always returns from the same side, Nadal will serve to the other side.
- Any situation in which any of the two players prefers one of the sides cannot be a Nash equilibrium.
- Conclusion:
  - Nadal must serve in a way that Federer does not prefer one side to the other.
  - Federer must return in a way that Nadal does not prefer one side to the other.

Nadal vs. Federer

- Say both play L-R 50-50 %.
- From Nadal's perspective, Federer plays 50-50:
  - If Nadal plays L he wins $\frac{1}{2}50 + \frac{1}{2}90 = 70\ \%$ of the times.
  - If Nadal plays R he wins $\frac{1}{2}70 + \frac{1}{2}50 = 60\ \%$ of the times.
- Nadal does not want to randomize between R and L, he prefers to play Left.
- In this situation, Federer does not want to randomize either.
- How is the situation in which both want to randomize?

Nadal vs. Federer. Computing the NE using indifference

- Let q be the probability with which Federer plays L (left), let's see what will Nadal do:

$$u_{\text{Nadal}}(L|q) = 0.5q + 0.9(1-q) = 0.9 - 0.4q,$$
$$u_{\text{Nadal}}(R|q) = 0.7q + 0.5(1-q) = 0.5 + 0.2q.$$

- For Nadal to be undecided we must have:

$$0.9 - 0.4q = 0.5 + 0.2q.$$
$$q = \frac{2}{3}.$$

Nadal vs. Federer. Computing the NE using indifference

- Let *p* be the probability with which Nadal plays *L* (left), let's see what will Federer do:

$$u_{FEDERER}(L|p) = 0.5p + 0.3(1-p) = 0.3 + 0.2p$$
$$u_{FEDERER}(R|p) = 0.1p + 0.5(1-p) = 0.5 - 0.4p$$

- For Federer to be undecided we must have:

$$0.3 + 0.2p = 0.5 - 0.4p$$
$$p = \frac{1}{3}$$

- This is how we write the equilibrium:

(1/3[L]+2/3[R], 2/3[L]+1/3[R])

Nadal vs. Federer. Computing the NE using the best reply

- Let's go back to Nadal's utilities:

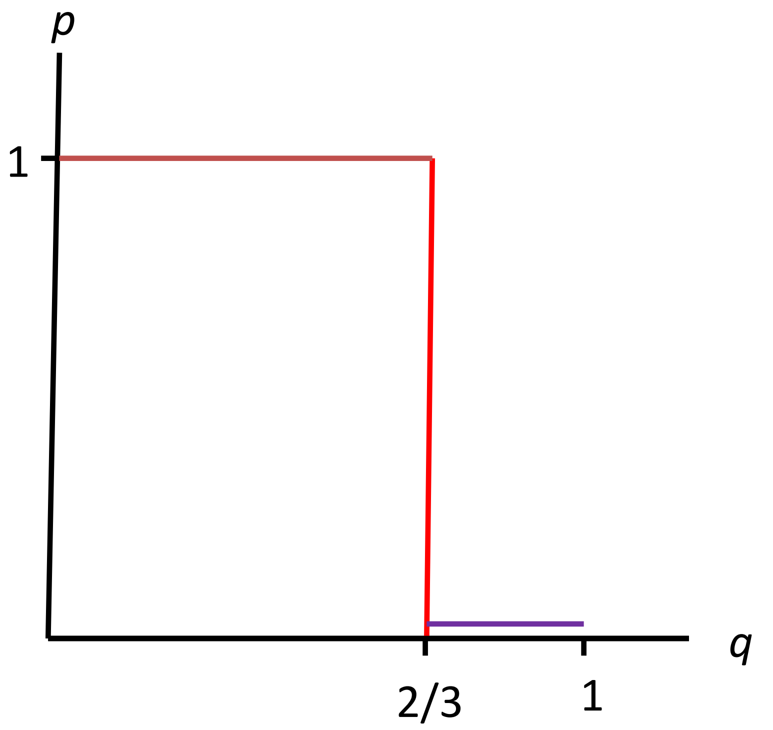
$$u_{\text{Nadal}}(L|q) = 0.9 - 0.4q,$$
$$u_{\text{Nadal}}(R|q) = 0.5 + 0.2q.$$

- If $q = \frac{2}{3}$ $\quad 0.9 - 0{,}4q = 0.5 + 0.2q$

  Any $p \in BR_{Nadal}(q = \frac{2}{3})$
- If $q > \frac{2}{3}$ $\quad 0.9 - 0.4q < 0.5 + 0.2q,$

  $(p = 0) = BR_{Nadal}(q > \frac{2}{3}),$
- If $q < \frac{2}{3}$ $\quad 0.9 - 0.4q > 0.5 + 0.2q,$

  $(p = 1) = BR_{Nadal}\left(q < \frac{2}{3}\right).$



Nadal vs. Federer. Computing the NE using the best reply

- Let's go back to Federer's utilities:

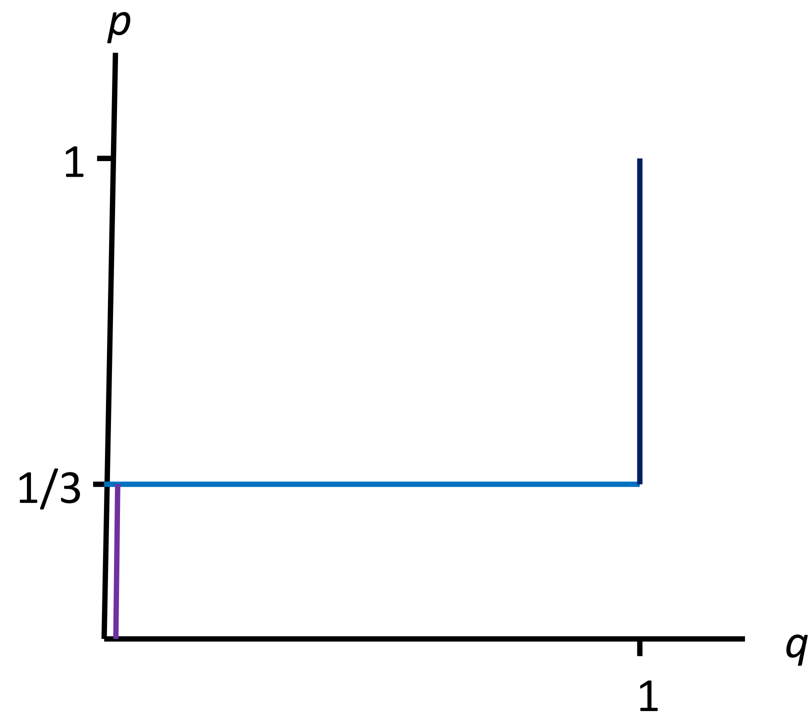
$$u_{Federer}(L|p) = 0.3 + 0.2p,$$
$$u_{Federer}(R|p) = 0.5 - 0.4p.$$

- If $p = \frac{1}{3}$ $\quad 0.3 + 0.2p = 0.5 - 0.4p$

  Any $q \in MR_{Federer}(p = \frac{1}{3})$
- If $p < \frac{1}{3}$ $\quad 0.3 + 0.2p < 0.5 - 0.4p$

  $(q = 0) = MR_{Federer}(p < \frac{1}{3}),$
- If $p > \frac{1}{3}$ $\quad 0.3 + 0.2p > 0.5 - 0.4p$

  $(q = 1) = MR_{Federer}\left(p > \frac{1}{3}\right).$



Nadal vs. Federer. Computing the NE using the best reply

- Nadal's best reply correspondence must tell us his best strategy for every possible mixed strategy by Federer:

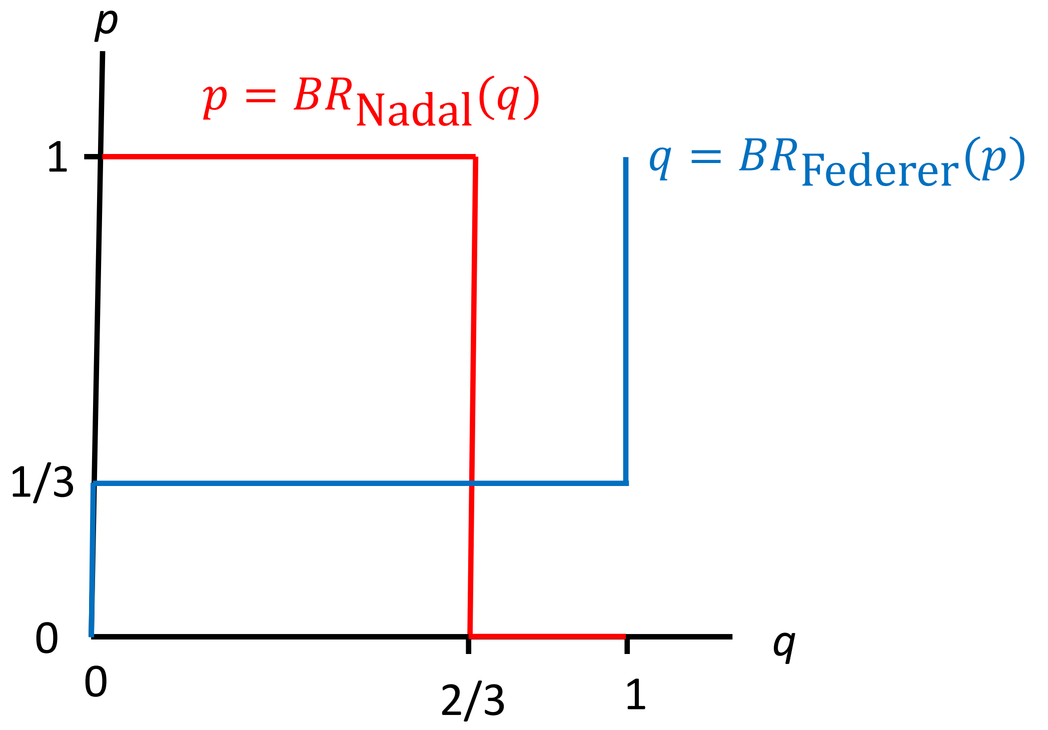
$$q < \frac{2}{3} \rightarrow \; BR_{\text{Nadal}} = L \qquad (p = 1),$$
$$q = \frac{2}{3} \rightarrow \; BR_{\text{Nadal}} = p \in [0,1],$$
$$q > \frac{2}{3} \rightarrow \; BR_{\text{Nadal}} = R \qquad (p = 0).$$

- Similarly, compute Federer's BR:

$$p < \frac{1}{3} \rightarrow \; BR_{\text{Federer}} = R \qquad (q = 0),$$
$$p = \frac{1}{3} \rightarrow BR_{\text{Federer}} = q \in [0,1],$$
$$p > \frac{1}{3} \rightarrow \; BR_{\text{Federer}} = L \qquad (q = 1).$$

Nadal vs. Federer. Computing the NE using the best reply

- Let's see the BR correspondence in a graph:



Domination with mixed strategies

- A pure strategy may be dominated by a mixed strategy.

	X	Y	Z
A	1, 1	2, 2	1, 4
B	2, 2	1, 1	2, 0
C	4, 3	0, 2	4, 1

- Let us see that strategy B is dominated by $q[A] + (1-q)[C]$ con $q \in \left(\frac{1}{2}, \frac{2}{3}\right)$.

Domination with mixed strategies

- Let us see if a mixed strategy with support AC dominates strategy B for player 1.

- A mixed strategy with support AC is of the form $q[A] + (1-q)[C]$.

- For this mixed strategy to dominate B, we must have
  $u_1(q[A] + (1-q)[C], s_2) > u_1(B, s_2)$ for all s_2:

  for $s_2 = x$: $\quad q + 4(1-q) > 2,$
  for $s_2 = y$: $\quad 2q > 1,$
  for $s_2 = z$: $\quad q + 4(1-q) > 2.$

- From there it follows that if $q \in \left(\frac{1}{2}, \frac{2}{3}\right)$, then $q[A] + (1-q)[C]$ dominates B.