Static games

2. Solution Concepts

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Normal form games (reminder)

- Static or simultaneous game: Each player takes their action without knowing the choice the others make.
- Elements of the simultaneous game:
  a) Players set: N = {1, . . . , n}.
  b) Strategies (actions) set S_i for each player. $S = X_{i=1}^{n} S_i$.
  c) (Expected) utility function over each of the strategy profiles. Thus, $u_i : S \rightarrow R$ for each player *i*.
- A normal form game is a triple (N, S, u).

How to play

- We will address the issue of how a perfectly rational agent would play:
  - This is a first step toward more realistic situations (not in this course)
  - This is a normative approach (not always descriptive)
- We will examine two different solution concepts:
  - Rationalizable strategies
  - Nash equilibrium

Dominated strategies

- Informally, a player's strategy is dominated if there exists another strategy that gives the player a higher utility regardless of the decision made by the other players.
- A rational player will never use a dominated strategy, as that behavior is inconsistent with utility maximization.

Dominated strategies

Formally they are defined as follows:

A strategy $s_i \in S_i$ is dominated if there is another strategy $t_i \in S_i$ such that $\forall s_{-i} \in S_{-i}$ we have:

$$u_i(s_i, s_{-i}) < u_i(t_i, s_{-i})$$

Note: sometimes it is called strictly dominated strategy, but the “strictly” part is just an emphasis.

Dominated strategies. Example

The prisoner's dilemma

		Prisoner 2	
		C	D
Prisoner 1	C	-1, -1	-5, 0
	D	0, -5	-4, -4

- For any of the prisoners, the strategy Cooperate (C) is dominated by the strategy Defeat (D).
- After the elimination of strategies C, we are left with (D,D) as the solution of the game.

Rationalizable strategies

- Let us have the game (N, S, u) and eliminate all dominated strategies.
- We will have a new game (N, S^1, u).
- In this new game we can proceed again to eliminate dominated strategies to obtain the game (N, S^2, u).
- We continue iteratively until no more strategies can be eliminated.
- If we stop at S^k, this will be the set of rationalizable strategies.

Rationalizable strategies

- The order of elimination of dominated strategies does not affect the result.
- Careful!: Weakly dominated strategies cannot be eliminated to find the set of rationalizable strategies.

Rationalizable strategies. Example

- Consider the game

	B1	B2	B3
A1	1, 1	0, 0	-1, 0
A2	0, 0	0, 6	10, -1
A3	2, 0	10, -1	-1, -1

- B3 is dominated by B1:

	B1	B2
A1	1, 1	0, 0
A2	0, 0	0, 6
A3	2, 0	10, -1

- Both A1 and A2 are dominated by A3.

Rationalizable strategies. Example

- After the last iteration we have:

	B1	B2
A3	2, 0	10, -1

- B2 is dominated by B1, so that {(A3, B1)} is the set of rationalizable strategies.
- In this example we found just one profile in the set, but there may be many.

More on domination

- A strategy $s_i \in S_i$ is (strictly) dominated if there exists another strategy $t_i \in S_i$ such that $\forall s_{-i} \in S_{-i}$ we have:

$$u_i(s_i, s_{-i}) < u_i(t_i, s_{-i})$$

(we will say that t_i dominates s_i).

- A strategy $s_i \in S_i$ is weakly dominated if there exists another strategy $t_i \in S_i$ such that $\forall s_{-i} \in S_{-i}$ we have:

$$u_i(s_i, s_{-i}) \leq u_i(t_i, s_{-i})$$

with $u_i(s_i, r_{-i}) < u_i(t_i, r_{-i})$ for some $r_{-i} \in S_{-i}$.

(we will say that t_i weakly dominates s_i).

- A strategy is dominant if it dominates all other strategies.
- A strategy is weakly dominant if it weakly dominates all other strategies.
- N.B.: Each of these strategies is defined for a player. There is not such a thing as a vector of strategies $s \in S$ being dominated or dominant.

Nash equilibrium (intuitions)

- The concept of Nash equilibrium identifies the strategy profiles for which no player has any incentive to deviate if he thinks that the other players will play according to the equilibrium.
- Each player is playing their best strategy given the play of the rest.
- No player has an incentive to unilaterally change their strategy.

Nash equilibrium

- Definition. A Nash equilibrium (NE) of a normal form game G is a strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ such that for each player *i* and each strategy $s_i \in S_i$ we have:

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

- Interpretations of the Nash equilibrium:
  - It is a self-sustained norm: once accepted, no player has any incentive not to follow it.
  - It is a profile of self-confirming expectations: if all players expect others to behave as prescribed, then these actions are the result of the players' behavior.

Nash equilibrium and razionalizable strategies

- A strictly dominated strategy can never be part of a Nash equilibrium.
- The set of razionalizable strategies includes all the Nash equilibria of the game.
- In a game where each player has only one rationalizable strategy, the set of rationalizable strategies constitutes the only Nash equilibrium.

Nash equilibrium. Calculation

- In a Nash equilibrium each player responds in the best way they can to the strategies of the other players.

- For each player $i \in N$ and each strategy profile for the rest of the players, $s_{-i} \in S_{-i}$, we identify the strategy (or strategies) that maximizes the utility of Player *i*, $BR_i(s_{-i})$. We will refer to $BR_i(s_{-i})$ as the best reply (or best response) by Player *i* against profile s_{-i}.

- This interpretation allows us to redefine the concept of Nash equilibrium as the solution of a system. If, for instance, *N* = 2, the NE, $s^* = (s_1^*, s_2^*)$, solves the system:

$$s_1 \in MR_1(s_2)$$
$$s_2 \in MR_2(s_1)$$

Nash equilibrium. Calculation

The technique to solve the system depends on the specific game:

- If the Best Replies are functions, the problem is reduced to solve a system of equations (it will be the case in some games where strategies are defined by a continuous variable).

$$s_1 = BR_1(s_2)$$
$$s_2 = BR_2(s_1)$$

- In other games we will look for the best replies of each player against the other players' choices. After observing which best replies satisfy the system we find the equilibrium. (This may be hard.)
- In games that can be represented in a matrix form, we can mark the best reply for each player against the others' choices, underlining the corresponding payoff. Those entries of the matrix where we underlined all payoffs define a Nash equilibrium.

Next we will see some examples of 3. Latter we will see examples of 1. and 2.

Nash equilibrium. Examples

Coordination

		Player 2	
		I_2	D_2
Player 1	I_1	$\underline{1}$, $\underline{1}$	0, 0
	D_1	0, 0	$\underline{1}$, $\underline{1}$

$$NE = \{(I_1, I_2), (D_1, D_2)\}$$

Nash equilibrium. Examples

Battle of the sexes

		Player 2	
		F_2	B_2
Player 1	F_1	$\underline{2}$, $\underline{1}$	0, 0
	B_1	0, 0	$\underline{1}$, $\underline{2}$

$$NE = \{(F_1,F_2), (B_1,B_2)\}$$

Nash equilibrium. Examples

Prisoner's Dilemma

		Player 2	
		C_2	D_2
Player 1	C_1	-1, -1	-4, 0
	D_1	0, -4	-3, -3

$$NE = \{(D_1, D_2)\}$$

Nash equilibrium. Examples

- Matching pennies

		Player 2	
		H	T
Player 1	H	1, -1	-1, 1
	T	-1, 1	1, -1

$$EN = \emptyset.$$