Game Theory
January 2016 Exam

Name **Group:**

You have two and a half hours to complete the exam

I. Short questions (5 points each)

I.1 What is the Bertrand's paradox? Why does it arise? How can it be overcome?

I.2 Define what is a subgame. Can we have a dynamic game, where two players alternate playing for three periods (e.g., first player A plays, then player B, and then lastly player A again), but we have only one subgame? If yes, give an example. If no, make an explanation why we should have more than one subgame.

I.3 Consider a repeated prisoner's dilemma in which the discount rate is close enough to 1. Formulate a strategy that can sustain cooperation in a subgame perfect Nash equilibrium.

I.4 What is a credible threat? Give a simple example of a non credible threat.

II. Problems (20 points each)

II.1 The government is considering the building of a highway between cities A and B. The cost of the highway is 100 millions, and will provide a revenue of 60 millions to City A and of 70 millions to City B. The government does not know the revenues of the highway to the cities, and proposes the following decision mechanism. Each city must declare a quantity, denoted by x_i, where $i = A, B$ and $0 \leq x_i \leq 100$. If $x_A + x_B \geq 100$ the highway will be built, then City A will pay $100 - x_B$, and City B will pay $100 - x_A$ (i.e., each city will pay the total cost minus the quantity declared by the other city), while the government will pay the difference until 100 millions. If $x_A + x_B < 100$ the highway will not be built, then each city will get zero and pay zero.

(a) Write down the profit function for each city. (3 points)
(b) Show that the pair of strategies in which each city declares a quantity equal to the own revenue by each city is a Nash equilibrium. (10 points)
(c) Show that, for each city, the strategy described in (b) is a (weakly) dominant strategy. (7 points)

(a) $u_A(x_A, x_B) = 60 - (100 - x_B) = x_B - 40$ if $x_A + x_B \geq 100$, and $u_A(x_A, x_B) = 0$ if $x_A + x_B < 100$.
$u_B(x_A, x_B) = 70 - (100 - x_A) = x_A - 30$ if $x_A + x_B \geq 100$, and $u_B(x_A, x_B) = 0$ if $x_A + x_B < 100$.

(b) $u_A(x_A = 60, x_B = 70) = 70 - 40 = 30$.
If City A deviates to $x'_A < 60$, it will get 30 or zero, depending on whether $x'_A + x_B \geq 100$ or $x'_A + x_B < 100$. If City A deviates to $x''_A > 60$, it will get 30. In either case, there is no gain.

$u_B(x_A = 60, x_B = 70) = 60 - 30 = 30$.
If City B deviates to $x'_B < 70$, it will get 30 or zero, depending on whether $x_A + x'_B \geq 100$ or $x_A + x'_B < 100$. If City B deviates to $x''_B > 70$, it will get 30. In either case, there is no gain.

No one gains after unilateral deviations from $(x_A = 60, x_B = 70)$, therefore it is a NE.

(c) If $60 + x_B \geq 100$: $u_A(x_A = 60, x_B) = x_B - 40$. In this case, any other strategy $x'_A \neq 60$ gives also $u_A(x'_A, x_B) = x_B - 40$ if $x'_A + x_B \geq 100$ and zero otherwise. (I.e., any other strategy gives the same or less.)

If $60 + x_B < 100$, which means that $x_B < 40$: $u_A(x_A = 60, x_B) = 0$. In this case, any other strategy $x'_A \neq 60$ gives also zero if $x'_A + x_B < 100$ or $x_B - 40 < 0$ if $x'_A + x_B \geq 100$. (I.e., any other strategy gives the same or less.)

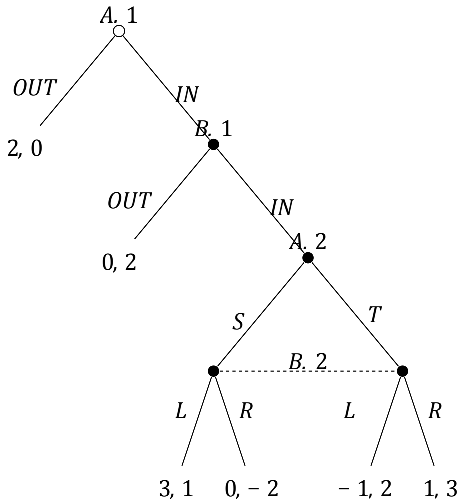
Similar for City B.

II.2 Two players, A and B, play the following game. First A must choose *IN* or *OUT*. If A chooses *OUT* the game ends, and A gets 2 and B gets 0. If A chooses *IN*, B observes this and then must choose *in* or *out*. If B chooses *out* the game ends, and B gets 2 and A gets 0. If B chooses *in* then they play the following simultaneous move game, where A chooses between rows:

	L	*R*
S	3, 1	0, -2
T	-1, 2	1, 3

(a) Draw the extensive form representation of this game. (3 points)
(b) Specify the information sets of each player. Also specify the subgames. How many subgames do we have? (5 points)
(c) Find all the pure strategy subgame perfect Nash equilibria of the game. (12 points)

(a)



(a) $I_A = \{A.1, A.2\}$, $I_B = \{B.1, B.2\}$. If we represent the subgames with their starting node, they are the games started with $A.1, B.1, A.2$, so there are 3 subgames.

(b) In the last subgame (the one represented by the matrix), there are two pure strategy equilibria (up; left) and (down; right). Each corresponds to a SPE of the whole game. The SPE are: [(OUT; up);(out; left)] and [(OUT; down);(in; right)].

There is one NE in mixed strategies in the subgame: $(\frac{1}{4}[S] + \frac{3}{4}[T], \frac{1}{5}[L] + \frac{4}{5}[R])$. The SPE contains it is $((OUT, \frac{1}{4}[S] + \frac{3}{4}[T]), (out, \frac{1}{5}[L] + \frac{4}{5}[R])$.

II.3 Anna and Bob are sharing a flat. Each of them has 2 hours of free time each day, which can be spent on sleeping or cleaning the flat. Each day, Anna and Bob simultaneously and independently allocate their free time between sleeping and cleaning. If during a day Anna spends x_A hours on cleaning the flat, while Bob spends x_B hours, then in that day Anna receives a payoff

$$u_A(x_A, x_B) = 2 - (x_A)^2 + (x_A + x_B),$$

while Bob receives a payoff

$$u_B(x_A, x_B) = 2 - (x_B)^2 + (x_A + x_B).$$

Here, $2 - (x_i)^2$ represents the utility that player $i \in \{A, B\}$ receives from sleeping, while $(x_A + x_B)$ represents the utility from cleanliness.

(a) Suppose Anna and Bob only share the flat for one day. Find all the Nash equilibria of the game. (5 points)
(b) Find the socially optimal allocation of time (that is, an allocation that maximizes the sum of Anna's and Bob's utilities). (4 points)
(c) Suppose Anna and Bob share the flat for three days. Each of them discounts the payoff with a discount factor δ. Find all the subgame perfect Nash equilibria of the game. (3 points)
(d) Suppose Anna and Bob share the flat for an indefinite number of days. Find the values of δ for which it is possible to find a subgame perfect Nash equilibrium in which the allocation of time is socially optimal. Find that equilibrium. (8 points)

(a) Max$_{xA}$ $u_A(x_A\,,\, x_B)$ subject to $x_A \epsilon [0,1]$
This gives $x_A = 0.5$. Similarly, $x_B = 0.5$, so the unique NE is $(x_A\,,\, x_B) = (0.5, 0.5)$.

(b) Max$_{(xA,xB)}$ $u_A(x_A\,,\, x_B) + u_B(x_A\,,\, x_B)$ subject to $x_A\,,\, x_B \,\epsilon\, [0,1]$
This gives the following first-order condition for Anna: $-2x_A + 2 = 0$, so $x_A = 1$. Similarly, $x_B = 1$.

(c) The stage game has a unique NE, so if the stage game is repeated finitely many times, there is a unique SPNE in which Anna and Bob each play $x_i = 0.5$ in each period.

(d) We can formulate the following trigger strategy for each player $i \epsilon \{A, B\}$: start by playing $x_i = 1$. In each period, play $x_i = 1$ if each player spent 1 hour cleaning the flat in the preceding day. Otherwise, play $x_i = 0.5$ forever.

Following this strategy gives each player a payoff of $2 – 1^2 + (1+1) = 3$ in each period, and a total payoff of:
$3 + 3\delta + 3\delta^2 + \ldots = 3/(1-\delta)$.

Deviating gives a maximum payoff of $2 – (0.5)^2 + (0.5 + 1) = 3.25$ in the period in which the player deviates. Subsequently, the player receives $2 – (0.5)^2 + (0.5 + 0.5) = 2.75$. So the total payoff from deviating is:
$3.25 + 2.75\delta + 2.75\delta^2 + \ldots = 3.25 + 2.75\delta/(1- \delta)$.

The trigger strategy described above is an NE if and only if $3/(1-\delta) \geq 3.25 + 2.75\delta/(1- \delta)$. I.e. iff $\delta \geq 0.5$.

It is also a SPNE:

In subgames after a history of (1,1) in every period, the game is identical to the original game and the trigger strategy requires top play as in the whole game. Because the trigger strategy is a NE in the whole game it is also a NE in these subgames.

In subgames after any play different from (1,1), the game is identical to the original game, but now the trigger strategy requires to play (0.5, 0.5) thereof, which is the repetition of the static NE, which is a NE.

II.4 In the island of Corruptia there are two political parties, XX and ZZ. Everyone knows that Party XX is of the tough type (D), but that ZZ can be tough (D) or weak (S) with probabilities p and $1-p$, respectively. Both parties must decide simultaneously what kind of elections campaign to conduct: a white-glove (GB) campaign or an aggressive one (A). If ZZ is of the tough type (D) the payoff matrix is:

XX \ ZZ	GB	A
GB	5, 5	3, 10
A	10, 0	2, 2

If ZZ is of the weak type (S) the matrix is:

XX \ ZZ	GB	A
GB	10, 5	5, 5
A	15, -5	0, -10

(a) Describe the above situation as a Bayesian game. (5 points)
(b) Find the set of Bayesian equilibria for the different values of p. (15 points)

(a) Players: {XX, YY}
Types of XX: {D_x}, types of ZZ: {D_z, S_z}
Conditional probabilities of types: $(p(D_X/D_Z) = 1)$, $(p(D_X/S_Z) = 1)$, $(p(D_Z/D_X) = p, p(S_Z/D_X) = 1-p)$.
Strategies for XX = {GB,A}. Strategies for ZZ = {(GB,GB) (GB,A) (A,GB) (A,A)}, wher the first component indicates the action that ZZ chooses when it is of type C, and the second component is the action of type S.

(b) Analyze the payoff matrix when ZZ is of type, and observe that party ZZ will chose A, as it strictly dominates GB. No domination occurs for the other type. Thus, out of the 4 strategies for ZZ whe only need to consider (A,GB) and (A,A).

To find the Bayesian equilibria, first find ZZ's best replay against XX's strategies:
BR_{ZZ}(GB) = (A,GB) and (A,A)
BR_{ZZ}(A) = (A,GB)

Now find XX's best reply against ZZ's strategies (A,GB) and (A,A). For that, we need to compute the payoffs:

$$\begin{array}{ccc} XX & (A,GB) & (A,A) \\ GB & 3p+10(1-p) & 3p+5(1-p) \\ A & 2p+15(1-p) & 2p+0(1-p) \end{array}$$

We see that when ZZ chooses (A, GB), the indifference between GB and A for XX occurs at:
$10-7p = 15-13p \leftrightarrow 6p = 5 \leftrightarrow p = 5/6$
If ZZ chooses (A,A) we have $5-2p > 2p$ for all p. Therefore MR_{XX}(A,A) = GB

$$MR_{XX}(A,GB) = \begin{cases} A\ si\ p < 5/6 \\ GB\ y\ A\ si\ p = 5/6 \\ GB\ si\ p > 5/6 \end{cases}$$

If $p < 5/6$ Bayesian equilibria (BE): {(A, (A,GB)); (GB,(A,A))}
If $p = 5/6$ BE: {(A, (A,GB)); (GB,(A,GB); (GB,(A,A))}
If $p > 5/6$ BE: {(GB, (A,GB)); (GB,(A,A))}
The Bayesian equilibrium (GB;(A,A)) does not depend on the value p. Thus (GB;(A,A)) is always a BE.