I	II.1	II.2	II.3	II.4	Total

Game Theory
Exam February 2021

Name: **Group:**

You have two and a half hours to complete the exam. No calculators or electronic devices are permitted.

I Short questions (5 points each)

I.1 Provide an example of a simultaneous game with both cooperation and conflict among players. Show the sources for cooperation and conflict.

I.2 Give an example of a game that includes a non-credible threat.

I.3 Give an example of a game that includes a non-credible promise.

I.4 Two players play an alternating offers bargaining game with 4 rounds and 1 unit of surplus to share. There is no discount and Player 1 makes the first offer. Which is the outcome in a SPNE?

I.1 See class notes.

II.2 See class notes.

II.3 See class notes.

II.4 Round 4: Player 2 makes the offer and gets everything. Payoffs in this subgame are (0, 1).
Round 3: Player 1 makes the offer. Whatever she does, Player 2 gets everything by rejecting and going to Round 4. Payoffs in this subgame are (0, 1).
Round 2: Player 2 makes the offer. He demands everything. Whatever Player 1 does, Player 2 gets everything now or later. Payoffs in this subgame are (0, 1).
Round 1: Player 1 makes the offer. Whatever she does, Player 2 gets everything by rejecting and going to the next round. Payoffs in this subgame, which is the whole game are (0, 1). This is the outcome of the game.

II. Problems (20 points each)

II.1 If a teacher invests x_1 units of time into preparing the lecture and a student invests $x_2$units of time into studying for the lecture, their utilities are the following:

$$u_1(x_1, x_2) = (42 + x_2)x_1 - 4x_1^2$$

$$u_2(x_1, x_2) = (42 + x_1)x_2 - 4x_2^2$$

That is, each one exerts benefits on the other one by preparing well for the lecture. However, preparation is costly for both.

(a) (5 points) Assuming that both take their decisions simultaneously and independently, without knowing the decision of the other agent, compute the best responses and draw them in a graph.
(b) (5 points) Compute the Nash equilibria and calculate the utilities of the players in them.
(c) (5 points) Could the teacher and the student achieve a higher utility if they could agree on the number of hours they spend on preparing the lectures? Hint: find the Pareto optimal symmetric time assignment.
(d) (5 points) Why the solution of the previous part is not reasonable if the teacher and the student cannot commit to their agreement?

(a) The first order condition is $42 + x_2 - 8x_1 = 0$ So, the best response of player 1 is

$$BR_1(x_2) = \frac{42 + x_2}{8}$$

By symmetry, the best response of player 2 is

$$BR_2(x_1) = \frac{42 + x_1}{8}.$$

(b) Solving the system of equations formed by de best reply functions one finds $x_1 = x_2 = 6$. The utilities of the players are $u_1(6,6) = u_2(6,6) = 144$.

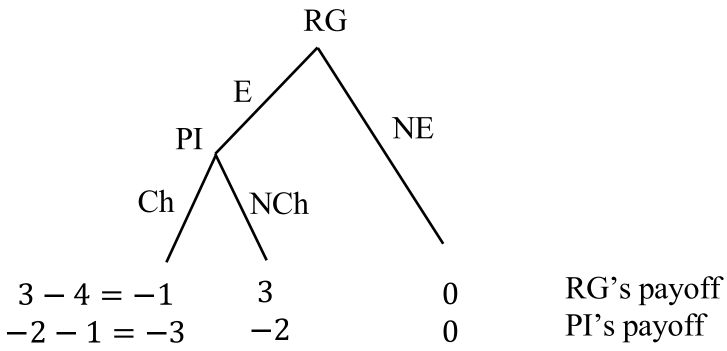
(c) A symmetric assignment means $x_1 = x_2 = x$.
Then $\max_x u_i(x) = (42 + x)x - 4x^2$ implies $x = x_1 = x_2 = 7$, with $u_1(7,7) = u_2(7,7) = 147$.

(d) The allocation $x_1 = x_2 = 7$ is not a NE. If, for example, Player 1 knows that $x_2 = 7$, then he would choose $x_1 = BR_1(7) = \frac{42+7}{8} = 6.125 \neq 7$. The same reasoning applies to Player 2.

II.2 RG Corporation produces OLED screens that Pear Inc. uses in its smartphones. RG considers entering the smartphone market, where it projects a profit of 3; however, Pear Inc. would lose 2 in profits if a new competitor arises. To prevent this Pear Inc. may threat to change its supplier of OLED screens which would cause a loss of 4 to RG and of 1 to Pear Inc. (in addition of what they get if RG enters). The threat is only carried out if RG enters the market.

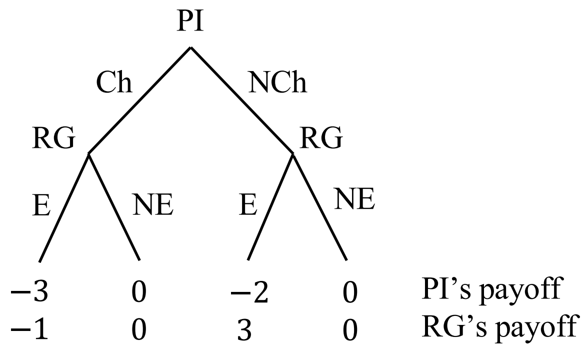
(a) (8 points) Draw a game and find the SPNE in case RG chooses first and Pearl Inc. chooses second after observing RG's choice.
(b) (8 points) Draw a game and find the SPNE if the order is reversed.
(c) (4 points) Comment on the differences in SPNE.

(a)



SPNE = (E, NCh).

(b)



SPNE = (Ch, (NE, E)).

(c) The firm playing first can commit to a strategy and make the outcome more favorable to it. Notice that payoffs are (3,-2) and (0, 0) in the respective equilibria.

II.3 Consider the dynamic game that consists on playing the following prisoners dilemma on a first stage:

		Player B C	Player B N
Player A	C	-4 , -4	-1 , -5
	N	-5 , -1	-2 , -2

Then, in stage 2, the same players play the following "choice of standards" game after observing what they played at stage 1 and after collecting the respective payoffs:

		Player B C	Player B N
Player A	C	3 , 3	0 , 0
	N	0 , 0	1 , 1

(a) (2 points) Find the stage Nash equilibria in pure strategies of the two different games above.
(b) (9 points) Find the minimum discount rate δ such that there exists a SPNE in which players play (N, N) in the first period. For this question, restrict attention to pure strategies.
(c) (9 points) Answer question (b) in the case in which we allow for mixed strategies.

(a) Pure strategies NE in the first stage game: (C, C).
Pure strategies NEa in the second stage game: (C, C) and (N, N).

(b) Strategy *S*:
-Play (N, N) in the first stage.
-Play (C, C) in the second stage if (N, N) was played in the first stage.
-Play (N, N) in the second stage if (N, N) was not played in the first stage.

For the strategy to be a SPNE we need $u_i(S) \geq u_i(\text{Deviation}_i, S_j)$, where $u_i(S) = -2 + 3\delta$.

In the second stage there are not profitable deviations, as Strategy S prescribes a Nash equilibrium in each subgame of the second stage. Thus, the best deviation by Player *i* is to play C in the first stage. Then payoff in this deviation is $u_i(\text{Deviation}_i, S_j) = -1 + 1\delta$.
From there we need $-2 + 3\delta \geq -1 + 1\delta$, which is true for all $\delta \geq \frac{1}{2}$.

(c) Mixed strategies NEa in the second stage game: $\left(\frac{1}{4}[\text{C}] + \frac{3}{4}[\text{N}], \frac{1}{4}[\text{C}] + \frac{3}{4}[\text{N}]\right)$, with payoffs $\left(\frac{3}{4}, \frac{3}{4}\right)$.

Strategy S':

-Play (N, N) in the first stage.
-Play (C, C) in the second stage if (N, N) was played in the first stage.
-Play $\left(\frac{1}{4}[\text{C}] + \frac{3}{4}[\text{N}], \frac{1}{4}[\text{C}] + \frac{3}{4}[\text{N}]\right)$ in the second stage if (N, N) was not played in the first stage.
Now, $u_i(S) \geq u_i(\text{Deviation}_i, S_j)$ implies $-2 + 3\delta \geq -1 + \frac{3}{4}\delta$, which is true for all $\delta \geq \frac{4}{9}$.

II.4 Argentina and Uruguay are trying to clean the Río de la Plata. Each country will clean from its side. This will cause that at the time of cleaning, each country will pass part of its residuals to the other side. As public resources are limited, if a country dedicates a proportion s_i of its personnel from the environmental office, there will be (1- s_i) left for other important projects. The preferences of the Uruguayan government, known to both countries, are:

$$U_U(s_U, s_A) = 2s_U - \frac{s_U s_A}{2} - \frac{s_U^2}{2} + (1 - s_U).$$

On its side, Argentina just change government, and its preferences are not well known to Uruguay. What it is known is that they are one of the following, either:

$$U_A(s_A, s_U) = 2s_A - s_A s_U - \frac{s_A^2}{2} + (1 - s_A),$$

meaning that they care a lot for residuals coming from the Uruguayan side, or:

$$U_A(s_A, s_U) = 2s_A - \frac{s_A s_U}{3} - \frac{s_A^2}{2} + (1 - s_A),$$

meaning that they do not care so much. The probabilities of these preferences are both ½.

(a) (5 points) Write the problem as a Bayesian game, showing all the elements.
(b) (6 points) Write the problem faced by the Argentinian government in the cases (i) it is highly bothered by the residuals coming from the Uruguayan side and (ii) it is not bothered that much. In both cases find the best reply functions.
(c) (6 points) Write the problem faced by the Uruguayan government and find its best reply function.
(d) (3 points) Find the Bayesian equilibria.

(a) Players = {A, U}. Types for A: $T_A = \{A^1, A^2\}$, types for U: $T_U = \{U\}$.
Beliefs: $(p(U|A^1) = 1)$, $(p(U|A^2) = 1)$, $\left(p(A^1|U) = \frac{1}{2}, p(A^2|U) = \frac{1}{2}\right)$.
Actions: $A_{A^1} = A_{A^2} = A_U = \{s_i \in [0,1]\}$.
Payoff functions = utility functions above.

(b)

$$\underset{s_A^1}{Max}\ 2s_A^1 - s_A^1 s_U - \frac{{s_A^1}^2}{2} + (1 - s_A^1) \text{ gives } 2 - s_U - s_A^1 - 1 = 0, \text{ and } s_A^1 = 1 - s_u.$$

$$\underset{s_A^2}{Max}\ 2s_A^2 - \frac{s_A^2 s_U}{3} - \frac{{s_A^2}^2}{2} + (1 - s_A^2) \text{ gives } 2 - \frac{s_U}{3} - s_A^2 - 1 = 0 \text{ and } s_A^2 = 1 - \frac{s_u}{3}.$$

(c)

$$\underset{s_U}{Max}\ \frac{1}{2}\left(2s_U - \frac{s_A^1 s_U}{2} - \frac{s_U^2}{2} + (1 - s_U)\right) + \frac{1}{2}\left(2s_U - \frac{s_A^2 s_U}{2} - \frac{s_U^2}{2} + (1 - s_U)\right) \text{ gives } 2 - \frac{(0.5s_A^1 + 0.5s_A^2)}{2} - s_U - 1 = 0 \text{ and } s_U = 1 - \frac{0.5s_A^1 + 0.5s_A^2}{2}.$$

(d) Solving the system formed with the three reaction functions we get:

EBN = $\{s_A^1 = \frac{1}{4}, s_A^2 = \frac{3}{4}, s_U = \frac{3}{4}\}$.