WORKSHEET 5: Integration

1. (*) Calculate the following integrals:

$a)\ \int \frac{x^2+x+1}{x\sqrt{x}}dx$ $\quad b)\ \int xe^{-2x}dx$ $\quad c)\ \int sin^{14}x\ \cos x\ dx$

$d)\ \int (x+1)(2-x)^{1/3}dx$ $\quad e)\ \int \frac{x^4}{1+x^5}dx$ $\quad f)\ \int (1+\frac{1}{x})^3\frac{1}{x^2}dx$

$g)\ \int sin^3 x\ dx$ $\quad h)\ \int xe^{ax^2}\ dx$ $\quad i)\ \int \frac{1}{3+x^2}dx$

$j)\ \int \frac{\sqrt{x-1}}{1+\sqrt[3]{x-1}}dx$ $\quad k)\ \int \frac{x}{\sqrt{16-x^2}}dx$ $\quad l)\ \int x^4 \ln x\ dx$

$m)\ \int \frac{dx}{\sqrt[4]{x^3}-\sqrt{x}}$ $\quad n)\ \int (\ln x)^2 dx$ $\quad ñ)\ \int \frac{40x}{(x-1)^{40}}dx$

$o)\ \int \frac{4x+6}{(x^2+3x+7)^3}dx$ $\quad p)\ \int \frac{2x-6}{(x-2)^2}dx$ $\quad q)\ \int \frac{x^2+1}{x^3-4x^2+4x}dx$

$r)\ \int \frac{2x+1}{x^3+6x}dx$ $\quad s)\ \int \frac{1}{\frac{x^2}{2}-2x+4}dx$ $\quad t)\ \int \frac{x^4}{x^4-1}dx$

Solution:

a) $\frac{2}{3}x^{3/2}+2x^{1/2}-2x^{-1/2}+C$

b) $xe^{-2x}/2-\frac{1}{4}e^{-2x}+C$

c) $(sin^{15}x)/15+C$

d) $(-3)\frac{3}{4}t^{4/3}+\frac{3}{7}t^{7/3}+C$

e) $\frac{1}{5}\ln(1+x^5)+C$

f) $(1+\frac{1}{x})^4/4+C$

g) $-\cos x+(\cos^3 x)/3+C$

h) $\frac{1}{2a}\ e^{ax^2}+C$

i) $\frac{\sqrt{3}}{3}\arctan(\frac{x\sqrt{3}}{3})+C$

j) $6[\frac{1}{7}\sqrt[6]{(x-1)^7}-\frac{1}{5}\sqrt[6]{(x-1)^5}+\frac{1}{3}\sqrt[6]{(x-1)^3}-\sqrt[6]{(x-1)}+\arctan\sqrt[6]{x-1}]+C$

k) $-\sqrt{16-x^2}+C$

l) $\frac{1}{5}x^5\ln x-\frac{1}{25}x^5+C$

m) $4\sqrt[4]{x}+4\ln(\sqrt[4]{x}-1)+C$

n) $x(\ln x)^2-2x\ln x+2x+C$

ñ) $40[(\frac{-1}{38})(x-1)^{-38}+(\frac{-1}{39})(x-1)^{-39})+C$

o) $2(x^2+3x+7)^{-2}\ /(-2)+C$

p) $2\ln(x-2)+\frac{2}{x-2}+C$

q) $\frac{1}{4}\ln x+\frac{3}{4}\ln(x-2)-\frac{5}{2}(x-2)^{-1}+C$

r) $\frac{-1}{12}\ln(x^2+6)+2\frac{\sqrt{6}}{6}\arctan(\frac{x\sqrt{6}}{6})+\frac{1}{6}\ln x+C$

s) $\arctan(\frac{x-2}{2})+C$

t) $x+\frac{1}{4}\ln(x-1)-\frac{1}{4}\ln(x+1)-\frac{1}{2}\arctan(x)+C$

2. How many different intersection points can two different primitives of the same function have?

   Solution: none.

3. (*) Let $f : [0,2] \longrightarrow \mathbb{R}$ be continuous, increasing in $(0,1)$, decreasing in $(1,2)$ and, also, satisfying that: $f(0) = 3$, $f(1) = 5$ and $f(2) = 4$. Between which values can we guarantee that $\int_0^2 f(x)\,dx$ is located?

   Solution: $7 \leq \int_0^2 f(x)\,dx \leq 10$.

4. (*) Certain company has determined that its marginal cost is $\frac{dC}{dx} = 4(1+12x)^{-1/3}$.

   Find the cost function if $C = 100$ when $x = 13$.

   Solution: $\frac{1}{3}(1+12x)^{2/3}.\frac{3}{2} + 100 - \frac{1}{2}(157)^{2/3}$

5. (*) Given that the marginal cost of producing x units is $x+5$ and the average cost has a minimum in $x = 4$, find the fixed costs of the firm.

   Solution: $C_f = 8$.

6. (*) Calculate $F'(x)$ in the following cases:

$$a) \int_x^{x^3} t\cos t\,dt \quad b) \int_1^{x^2} \sqrt{t^4+2t}\,dt \quad c) \int_1^{x^2} \left(t^2-2t+5\right)\,dt$$

   Solution:

   $a)$ $F'(x) = 3x^5\cos x^3 - x\cos x$

   $b)$ $F'(x) = \sqrt{x^8+2x^2}.2x$

   $c)$ $F'(x) = 2x^5 - 4x^3 + 10x$

7. Calculate $F'(x)$ in the following cases:

   (a) $\int_{-x}^{x^2} \tan^2 t\,dt$, supposing that $x^2 < \frac{\pi}{2}$.

   (b) $\int_{x^2}^{2x} f^2(2t)\,dt$, supposing that f is continuous.

   Solution:

   (a) $2x\tan^2 x^2 + \tan^2 x$

   (b) $2\ f^2(4x) - 2xf^2(2x^2)$

8. (*) What are the values of x where $F(x) = \int_{-3}^{x} \frac{t^2-4}{3t^2+1}dt$ has a local maximum or minimum?

   Solution: F reaches a local minimum in 2 and a local maximum in -2.

9. Let $F(x) = \int_{x^2}^{2x} f(t^2)dt$ be such that $f(1) = 1$, $f(2) = f(4) = 4$ and f is continuous. Calculate $F'(1)$.

   Solution: $F\,'(1) = 6$

10. (*) Calculate observing the symmetry of the functions:

$$a)\ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sin^{27}x \cos^{28} x\, dx \quad b)\ \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (\sqrt[3]{x^5 \cos 3x} + \cos \tfrac{x}{3} + \tan^3 x\,)dx$$

Solution:

(a) 0

(b) $6sin\frac{\pi}{9}$.

11. Let f be a function with period T, such that $\int_0^T f = b$. Find $\int_a^{a+nT} f$.

Solution: $\int_a^{a+nT} f = \int_0^{0+nT} f = nb$

12. (*) Find the area located between the following curves:

$$\begin{aligned} &a)\ f(x) = x^2 - 4x + 3,\ g(x) = -x^2 + 2x + 3 \\ &b)\ f(x) = (x-1)^3,\ g(x) = x - 1 \\ &c)\ f(x) = x^4 - 2x^2 + 1,\ g(x) = 1 - x^2 \end{aligned}$$

Solution:

(a) 9.

(b) $\frac{1}{2}$.

(c) $\frac{4}{15}$.

13. (*) Graph the functions $y = 2e^{2x}$ and $y = 2e^{-2x}$. Calculate the area located between those graphs and the lines $x = -1$ and $x = 1$.

Solution: $2(e^2 + e^{-2} - 2)$

14. Let $f : [1,3] \longrightarrow [2,4]$ be increasing, continuous and bijective such that $\int_1^3 f\, dx = 5$. Calculate $\int_2^4 f^{-1}(x)\, dx$

Solution: 5

15. (a) Given $f : [0,4] \to \mathbb{R}$, convex and increasing with values $f(0) = 0$, $f(2) = \alpha$, $f'(2) = \beta$, $f(4) = 16$. Estimate as a function of α and β, the value of $\int_0^2 f(x)dx$.

(b) Given $f : [0,4] \to \mathbb{R}$, concave and increasing with values $f(0) = 0$, $f(2) = \alpha$, $f'(2) = \beta$, $f(4) = 2$. Estimate as a function of α and β, the value of $\int_0^2 f(x)dx$.

Solution:

(a) $2(\alpha + \beta) < \int_0^4 f(x)dx < 2\alpha + 32$.

(b) $2\alpha < \int_0^4 f(x)dx < 2(\alpha - \beta) + 32$.

16. The sales of a product are given by the formula $S(t) = 10 + 5sin(\frac{\pi t}{6})$ where S is measured in thousands of units and time t in months. Calculate the average sales during the year $(0 \le t \le 12)$.

Solution: 10

17. Calculate:

$$a)\int_0^1 \frac{1}{\sqrt{x}}dx \quad b)\int_0^3 \frac{1}{x^3}dx \quad c)\int_1^\infty \frac{1}{x^2}dx$$
$$d)\int_1^\infty e^{-x}dx \quad e)\int_{-\infty}^\infty \frac{dx}{1+x^2} \quad f)\int_{-2}^4 \frac{dx}{x^2}$$

Solution:

a) 2

b) ∞

c) 1

d) 1

e) π

f) ∞

18. Calculate $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(1+x)}$

Solution: π