WORKSHEET 3 : Differentiation I

1. Find the points where the following functions have horizontal tangent.

   a) $f(x) = x^3 + 1$ b) $f(x) = 1/x^2$ c) $f(x) = x + sinx$
   d) $f(x) = \sqrt{x-1}$ e) $f(x) = e^x - x$ f) $f(x) = sinx + \cos x$

   a) x=0; b) never; c) $x = \pi + 2k\pi$; d) never; e) $x = 0$; f) $x = \frac{\pi}{4} + k\pi$.

2. (*)Prove that the tangent lines to the graphs of $y = x$ and $y = 1/x$ in their points of intersection are perpendicular to each other.

3. In what point is the tangent to the curve $y^2 = 3x$ pararell to the line $y = 2x$?

   The point of the curve is $(\frac{3}{16}, \frac{3}{4})$.

4. (*)Calculate the intersection point with the x axis of the tangent line to the graph of $f(x) = x^2$ in the point $(1, 1)$.

   The intersection point is $x = 1/2$.

5. Calculate a so that the tangent to the graph of $f(x) = a/x + 1$ in the point $(1, f(1))$ intersects the horizontal axis in $x = 3$.

   Therefore, the intersection point will be x=3 when a=1.

6. (*)Find the tangent and normal lines to $f(x) = \arctan\left(\frac{sinx}{1+\cos x}\right)$ in $x = 0$.

   Ecuation of the tangent line: $y - 0 = \frac{1}{2}(x - 0)$; ecuation of the normal line: $y - 0 = -2(x - 0)$.

7. Find the derivatives of the following functions.

   a) $f(x) = (sinx + \tan 3x)sin2x$ b) $f(x) = \dfrac{x\sqrt{x^2-1}}{2x+6}$
   c) $f(x) = 4x^{3/2}\cos 2x$ d) $f(x) = 5x\ln(8x + sin2x) + e^{\tan 5x}$

8. (*)Let $f(x) = 2[\ln(1 + g^2(x))]^2$. Using that $g(1) = g'(1) = -1$, calculate $f'(1)$.

   $f\,'(1) = 4\ln(2)$.

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9. (*)Using that $a^b = e^{b\ln a}$, differentiate $f(x) = x^{sinx}$ and $g(x) = (\sqrt{x})^x$.

   $f\,'(x) = x^{sinx}(\cos x. \ln x + sinx/x)$.

   $g'(x) = (\sqrt{x})^x(\ln x + 1)/2$.

10. (*)Let $f(x) = \ln(1 + x^2)$ and $g(x) = e^{2x} + e^{3x}$. Calculate $h(x) = f(g(x))$, $v(x) = g(f(x))$, $h'(0)$ and $v'(0)$.

    $h(x) = \ln(1 + e^{4x} + e^{6x} + 2e^{5x}), h'(0) = 4$

    $v(x) = (1 + x^2)^2 + (1 + x^2)^3, v'(0) = 0.$

11. Let $f : [-2, 2] \to [-2, 2]$ be continuous and bijective.

a) Suppose that $f(0) = 0$ and $f'(0) = \alpha$, $\alpha \neq 0$. Find $\left(f^{-1}\right)'(0)$.
b) Now suppose that $f(0) = 1$ and $f'(0) = \alpha$, $\alpha \neq 0$. Find $\left(f^{-1}\right)'(1)$.
c) Now suppose that $f(1) = 0$ and $f'(1) = \alpha$, $\alpha \neq 0$. Find $\left(f^{-1}\right)'(0)$.
a) $\left(f^{-1}\right)'(0) = \frac{1}{\alpha}$.
b) $\left(f^{-1}\right)'(1) = \frac{1}{\alpha}$.
c) $\left(f^{-1}\right)'(0) = \frac{1}{\alpha}$.

12. (*)Supposing that the following equations define y as a differentiable function of x, calculate y' in the given points:

    a) $x^3 + y^3 = 2xy$ in $(1,1)$.
    b) $sinx = x(1 + \tan y)$ in $(\pi, 3\pi/4)$.
    c) $x^2 + y^2 = 25$ in $(3,4), (0,5)$ and $(5,0)$.

    a) $y' = -1$.b) $y' = \frac{-1}{2\pi}$.c) the derivative in that point does not exist.

13. Calculate the derivative of the following functions showing where they are not differentiable.

    a)(*) $f(x) = \begin{cases} x^2 - 1 & \text{if } x \le 0 \\ 0 & \text{if } x > 0 \end{cases}$ b) (*)$g(x) = \begin{cases} 1/|x| & \text{if } x \le -2 \\ (x+2)^2 & \text{if } -2 < x \le 0 \\ 3 + sin(x + \frac{\pi}{2}) & \text{if } x > 0 \end{cases}$

    c) $h(x) = \begin{cases} \arctan^2 x & \text{if } x \le 0 \\ sin^3 x & \text{if } 0 < x \le 2\pi \\ sinx & \text{if } 2\pi < x \end{cases}$

    a) $f\,'(x) = \begin{cases} 2x & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$ Obviously, f is not differentiable in 0.

    b) $g\,'(x) = \begin{cases} 1/x^2 & \text{if } x < -2 \\ 2(x+2) & \text{if } -2 < x < 0 \\ \cos(x + \frac{\pi}{2}) & \text{if } x > 0 \end{cases}$ Obviously, f is not differentiable in -2 nor in 0.

    c) $h\,'(x) = \begin{cases} \frac{2\arctan x}{1+x^2} & \text{if } x \le 0 \\ 3sin^2 x.\cos x & \text{if } 0 \le x < 2\pi \\ \cos x & \text{if } 2\pi < x \end{cases}$ Obviously, f is not differentiable in 2π.

14. (*)Find a and b so that the function $f(x) = \begin{cases} 3x + 2 & \text{if } x \ge 1 \\ ax^2 + bx - 1 & \text{if } x < 1 \end{cases}$ is differentiable.

    f differentiable in 1 when $a = -3, b = 9$.

15. Apply the mean value theorem to f in the given interval and find the c values of the thesis of the theorem.

    a) $f(x) = x^2$ in $[-2,1]$ b) $f(x) = -2sinx$ in $[-\pi, \pi]$
    c) $f(x) = x^{2/3}$ in $[0,1]$ d) $f(x) = 2sinx + sin2x$ in $[0, \pi]$

16. Let $f : [a,b] \longrightarrow [a,b]$ be a continuous function in $[a,b]$ and differentiable in (a,b). Prove that, if $f\,'(x) \ne 1$ in (a,b), then f has a unique fixed point in $[a,b]$.

17. Prove that the function f has a unique fixed point.

    a) $f(x) = 2x + \frac{1}{2} sinx$ b) $f(x) = 2x + \frac{1}{2}\cos x$

18. (*)Let $f(x) = x^3 - 3x + 3$, $f : [-3,2] \to \mathbb{R}$. Determine the global extrema.

    The minimum is reached in -3 and the maximum is reached in -1 and in 2.

19. Let $f : [-5,5] \to \mathbb{R}$ such that f reaches the maximum in $x = 2$ and the minumum in $x = -3$. Let $g(x) = -f(-x)$. What can be said about the maximum and the minimum of g?