Information Advantage in Tullock Contests

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Abstract

We study the impact of an information advantage on the equilibrium payoffs and efforts in Tullock contests where the common value of the prize and the common cost of effort are uncertain. We show that if the cost of effort is linear then a player's information advantage is rewarded. For symmetric contests, we explicitly calculate the unique equilibrium and establish information invariance of the expected effort. We then study two-player contests with state-independent convex costs from the family $c(x) = x^{\alpha}$. Whereas players' expected costs of effort turns out to be the same regardless of their information asymmetry, in expectation a player with information advantage exerts no more effort, and wins the prize no more frequently, than his opponent. In classic Tullock contests (i.e., $\alpha = 1$), both players are shown to exert the same expected effort, which is larger when players have symmetric information than when one player has information advantage. Finally, we show that all-pay auctions do not necessarily provide better incentives to exert effort than do Tullock contests.

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1 Introduction

Tullock contests (see Tullock 1980) are perhaps the most widely studied models in the literature on imperfectly discriminating contests. In a Tullock contest a player's probability of winning the prize is the ratio of the effort he exerts and the total effort exerted by all players. This paper belongs to a relatively recent but growing strand of this literature that concerns Tullock contests with incomplete information. Specifically, we study Tullock contests in which the players' common value for the prize as well as their common cost of effort depend on the state of nature. Players have a common prior belief, but upon the realization of the state of nature (and before taking action) each player obtains some information pertaining to the realized state. The interim information endowment of each player at the moment of taking action is described by a σ-field of subsets of the state space, known as *events*: the player knows which events in his information field have occurred, and which have not, once the state is realized. The information fields may differ across players. This representation of players' uncertainty and information is natural, and encompasses the most general structures. In particular, it includes Harsanyi's model of Bayesian games.

In this setting, we show that Tullock contests reward information advantage: if some player i has more information about the uncertain parameters of the game than another player j, then the expected payoff of player i is greater than or equal to that of player j. This result holds for any two players with rankable information fields, regardless of the information endowments of the other players. Its proof relies on the proof of the theorem of Einy *et al.* (2002), showing that in any Bayesian Cournot equilibrium of an oligopolistic industry a firm's information advantage is rewarded.

We then proceed to study the impact of information advantage on effort. We identify a system of equations that all equilibria must satisfy. Using this system we establish a number of properties of the equilibria of these contests. First, we explicitly calculate the players' effort in *classic* Tullock contests in which players have symmetric information. It turns out the in equilibrium (which is unique, interior and symmetric), a player's expected effort is invariant to the level of information. Further, while the each player's expected effort decreases with the number of players, the expected total effort increases.

Next we derive properties of two-player contests in which the cost of effort is a convex function of the form $c(x) = x^\alpha$. We first show that regardless of the players' information, in any equilibrium their expected *costs of effort* coincide. Using this result, we show that the expected effort of the player with an information advantage is less or equal to that of his disadvantaged opponent; the former's ex-ante probability to win is also less or equal to that of the latter. The comparison is only strict for strictly convex costs. In fact, in classic Tullock contests, i.e., when $\alpha = 1$, both players exert the same expected effort regardless of their respective information endowments (that may not be comparable). But, when the information is comparable, a shift towards symmetry increases effort: the two players exert no less effort when they have the same information compared to the information advantage case. We also present examples showing that these results do not generalize to contests with more than two players.

Finally, we study whether all-pay auctions provide better incentives for players to exert effort than do Tullock contests. Einy *et al.* (2017) characterize the unique equilibrium of a two-player common-value all-pay auction, which is in mixed strategies, and provide an explicit formula that allows us to compute the players' total effort. We show that the sign of the difference between the total effort exerted by players in an all-pay auction and in a Tullock contest is ambiguous.

Relation to the literature

There is an extensive literature on Tullock contests under complete information. Baye and Hoppe (2003) have identified a variety of economic settings (rent-seeking, innovation tournaments, patent races), which are strategically equivalent to a Tullock contest. Skaperdas (1996) and Clark and Riis (1998) provide axiomatic characterizations of Tullock contests. Perez-Castrillo and Verdier (1992), Baye, Kovenock and de Vries (1994), Szidarovszky and Okuguchi (1997), Cornes and Hartley (2005), Yamazaki (2008) and Chowdhury and Sheremeta (2009) study existence and uniqueness of equilibrium. Skaperdas and Gan (1995), Glazer and Konrad (1999), Konrad (2002), Cohen and Sela (2005) and Franke *et al.* (2011) look into the effects of changes in the payoff structure on the behavior of players, and Schweinzer and Segev (2012) and Fu and Lu (2013) study optimal prize structures.

The study of Tullock contests under incomplete information is relatively sparse, however. Fey 2008 and Wasser 2011 study rent-seeking games under asymmetric information. Einy *et al.* (2015a) show that under standard assumptions Tullock contests with asymmetric information have pure strategy Bayesian Nash equilibria, although they neither characterize equilibrium strategies nor they study their properties.

More closely related to the topic of the present paper are the articles of Warneryd (2003), and Einy et al. (2016). Warneryd (2003) studies two-player Tullock contests in which the players' marginal cost of effort is constant and state-independent, and the value is a continuous random variable. In this setting, Warneryd considers the information structures arising when each player either observes the value, or has only the information provided by the common prior. Our results for two-player contests extend some of Warneryd's results to contests with the most general information structures. Moreover, we obtain results for two-player contests when the cost functions are not linear and when players information are not rankable, as well as for contests with more than two players. Additionally, we study the impact of information on payoffs and show that information advantage is rewarded in contests with any number of players and general information structures. Einy et al. (2016) also study the impact of information in Tullock contests, but their information is public and the attention is restricted to the symmetric information case.

As for the comparison of the outcomes generated by Tullock contests and all-pay auctions, Fang (2002), Epstein, Mealem and Nitzan (2011) study this issue under complete information, and Dubey and Sahi (2012) consider an incomplete information binary setting. Common-value first-price and second-price auctions have been studied by Einy *et al.* (2001, 2002), Forges and Orzach (2011), and Malueg and Orzach (2009, 2012), while all-pay auctions have been studied by Einy *et al.* (2017, 2017a) and Warneryd (2012).

The rest of the paper is organized as follows: Section 2 describes our setting. Section 3 studies the impact of information on payoffs, and Section 4 – its impact on efforts. Section 4 is dedicated to the question of whether Tullock contests or all-pay auctions are better in providing incentives to exert effort. Some technical proofs are relegated to the Appendix.

2 Common-Value Tullock Contests

A group of players $N = \{1, ..., n\}$, with $n \geq 2$, compete for a prize by exerting effort. Players' uncertainty about the value of the prize and the cost of effort is described by a probability space $(\Omega, \mathcal{F}, p)$, where Ω is the set of states of nature, $\mathcal{F}$ is a σ-field of subsets of (or events in) Ω, and p is a probability measure on $(\Omega, \mathcal{F})$ representing the players' common prior belief. Players' common value for the prize is an $\mathcal{F}$-measurable and integrable random variable

$V : \Omega \to \mathbb{R}_{++}$. Players' common cost of effort is given for all $(\omega, x) \in \Omega \times \mathbb{R}_+$ by $W(\omega)c(x)$, where $W : \Omega \to \mathbb{R}_{++}$ is an $\mathcal{F}$-measurable and integrable random variable, and $c : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable, strictly increasing and concave function satisfying $c(0) = 0$. We assume that the private information of every player is described by a σ-subfiled $\mathcal{F}_i$ of $\mathcal{F}$. This means that, for *any* event $A \in \mathcal{F}_i$, player i knows whether the realized state of nature is contained in A; in particular, if $\mathcal{F}_i$ is generated by a finite or countably infinite partition of Ω, then i knows the exact element of the partition containing the realized state.

A *common-value Tullock contest* (to which we will henceforth refer as a *Tullock contest*) starts by a move of nature that selects a state ω from Ω, of which every player i has partial knowledge (via $\mathcal{F}_i$). Then the players simultaneously choose their effort levels, $x = (x_1, ..., x_n) \in \mathbb{R}^n_+$. The prize is awarded to the players in a probabilistic fashion, using a contest contest success function $\rho : \mathbb{R}^n_+ \to \triangle^n$, where $\triangle^n$ is the n-simplex. Specifically, for $x \in \mathbb{R}^n_+$, if $\sum_{k=1}^n x_k > 0$, then the probability that player $i \in N$ receives the prize is

$$\rho_i(x) = \frac{x_i}{\sum_{k=1}^n x_k},$$

whereas if $\sum_{k=1}^n x_k = 0$, i.e., if no player exerts effort, then the prize is allocated according to some fixed probability vector $\rho(0) \in \triangle^n$. Hence, the payoff of player $i \in N$ is

$$u_i(\omega, x) = \rho_i(x)V(\omega) - W(\omega)c(x_i). \tag{1}$$

A Tullock contest defines a Bayesian game in which a pure strategy for player $i \in N$ is an $\mathcal{F}_i$-measurable integrable function $X_i : \Omega \to \mathbb{R}_+$, which describes i's choice of effort in each state of nature, conditional on his private information. We denote by S_i the set of strategies of player i, and by $S = \times_{i=1}^n S_i$ the set of strategy profiles. Given a strategy profile $X = (X_1, ..., X_n) \in S$ we denote by X_{-i} the profile obtained from X by suppressing the strategy of player i. Throughout the paper we restrict attention to pure strategies.

A strategy profile $X = (X_1, ..., X_n)$ is a *(Bayesian Nash) equilibrium* if for every $i \in N$ and every $X_i' \in S_i$,

$$E[u_i(\cdot, X(\cdot)] \geq E[u_i(\cdot, X_{-i}(\cdot), X_i'(\cdot))]; \tag{2}$$

or equivalently, if for every $i \in N$ and every $X_i' \in S_i$,

$$E[u_i(\cdot, (X(\cdot)) \mid \mathcal{F}_i] \geq E[u_i(\cdot, (X_{-i}(\cdot), X_i'(\cdot)) \mid \mathcal{F}_i] \tag{3}$$

almost everywhere on Ω, where $E[f \mid \mathcal{F}_i]$ denotes the conditional expectation of an $\mathcal{F}$-measurable random variable f with respect to the σ-field $\mathcal{F}_i$ – see Borkar (1995), section 3.1.

Einy *et al.* (2015a) provide conditions that imply the existence of equilibrium in the contests that we consider, at least when the information fields are generated by finite or countably infinite partitions of Ω.

Remark 1. *If X is a equilibrium then $\sum_{i\in N} X_i(\omega) > 0$ for almost every $\omega \in \Omega$.*

Proof. Let X be an equilibrium and assume, contrary to our claim, that there exists a positive-measure set $B \in \mathcal{F}$ such that $X_1 = ... = X_n = 0$ on B. Let i be a player for whom $\rho_i(0) \leq \frac{1}{2}$. Since X_i is $\mathcal{F}_i$-measurability there is $A_i \in \mathcal{F}_i$ such that $B \subset A_i$ and $X_i = 0$ a.e. on A_i. Let $\varepsilon > 0$, and consider a strategy $X_i' = \varepsilon 1_{A_i} + X_i 1_{\Omega\backslash A_i} \in S_i$. Then $\rho_i(X) \leq \rho_i(X_{-i}, X_i')$ on A_i, and $E[\rho_i(X_{-i}, X_i') \mid B] = 1$. Therefore by switching from X_i to X_i', player i's expected payoff remains unchanged on $\Omega\backslash A_i$ and increases on A_i by at least

$$\frac{1}{2}E[V \mid B]p(B) - c(\varepsilon)E[W \mid A_i]p(A_i),$$

which is positive for a sufficiently small ε, since $c(0) = 0$ and c is continuous at 0. Hence X_i' is a profitable deviation, contradicting that X is an equilibrium. ■

By Remark 1, the vector $\rho(0) \in \triangle^n$ used to allocate the prize when no player exerts effort does not affect the set of equilibria. Hence we may describe a Tullock contest by a collection $T = (N, (\Omega, , \mathcal{F}, p), \{\mathcal{F}_i\}_{i\in N}, V, W, c)$. Contests in which $W = 1_\Omega$ and $c(x) = x$ will be called *classic* Tullock contests.

3 Information Advantage and Payoffs

Our first result is concerned with the natural question of whether an information advantage is reflected in equilibrium payoffs. Formally, player $i \in N$ is said to have an information advantage over player $j \in N$ if $\mathcal{F}_i \supset \mathcal{F}_j$. Thus, the information of i on the realized state of nature is never less precise that that of j : whenever player j knows that the realized $\omega \in \Omega$ is contained in some $A \in \mathcal{F}_j$, there exists $B \subset A$, $B \in \mathcal{F}_i$, such that i knows that ω is contained in B.

Proposition 1 shows that an information advantage is rewarded in Tullock contests in which the deterministic component of the cost of effort c is a linear function: in these contests the expected payoff of a player is never below that of another player with less information. This result holds, in particular, in *classic* Tullock contests. Proposition 1 is proved by observing a formal equivalence between a Tullock contest and a Cournot oligopoly with asymmetric information, and by appealing to (the proof of) a result of Einy *et al.* (2002) which shows that the (Bayesian Cournot) equilibria of such industries have the desired property.

Proposition 1. *Assume that* $X = (X_1, ..., X_n)$ *is an equilibrium of a Tullock contest with* $c(x) = x$ *(i.e., the state-contingent marginal cost is constant), for which there exists* $a > 0$ *such that* $\sum_{j=1}^{n} X_j(\omega) \geq a$ *at almost every* $\omega \in \Omega$*. If some player* i *has an information advantage over some other player* j*, then* $E[u_i(\cdot, X(\cdot)] \geq E[u_j(\cdot, X(\cdot)]$.

Proof. Let $(N, (\Omega, \mathcal{F}, p), (\mathcal{F}_i)_{i \in N}, V, W, c)$ be a Tullock contest. For $X = (X_1, ..., X_n) \in S$ and $\omega \in \Omega$, the payoff of each player $i \in N$ may be written as

$$\begin{aligned} u_i(\omega, X(\omega)) &= \frac{X_i(\omega)}{\sum_{j=1}^{n} X_j(\omega)} V(\omega) - W(\omega) c(X_i(\omega)) \\ &= P(\omega, \sum_{j=1}^{n} X_j(\omega)) X_i(\omega)) - C(\omega, X_i(\omega)), \end{aligned}$$

where the functions $P, C : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ are defined as

$$P(\omega, x) = \frac{V(\omega)}{x}, \text{ and } C(\omega, x) = W(\omega) c(x). \tag{4}$$

Thus, if X is an equilibrium of the contests, then X is an equilibrium of the oligopolist industry $(N, (\Omega, \mathcal{F}, p), (\mathcal{F}_i)_{i \in N}, P, C)$, where P is the inverse market demand and C is the firms' cost function.

Einy *et al.* (2002) showed that information advantage is rewarded in any equilibrium of an oligopolist industry under certain conditions on the inverse demand function and costs. Some of the conditions are not satisfied, however, by the function P in (4). Fortunately, the proof of Einy *et al.* can be utilized in the present case too, provided it is shown that, for every $i \in N$,

$$E\left[1_{X_i > 0} \times \frac{d}{dx_i} u_i(\cdot, X(\cdot)) \mid \mathcal{F}_i\right] = 0. \tag{5}$$

(Equation (5) yields (2.6) on page 157 in Einy *et al.* (2002), from which point on their proof applies without change). We establish that equation (5) holds in the Appendix. ■

It is important to note that Proposition 1 does not involve any assumption about the information of the players whose information fields are *not* being compared: a player's information advantage over another player is rewarded regardless of the information endowments of the other players. That is, the conclusion of Proposition 1 holds whenever two players have rankable information.

Remark 2 shows that the qualification in Proposition 1 that the sum of equilibrium efforts is bounded away from zero holds under some general conditions. The proof of this remark is given in the Appendix.

Remark 2. *Let X be any equilibrium of a Tullock contest. If either*

(a) $N = \{1, 2\}$, $0 < \inf V \leq \sup V < \infty$, and $0 < \inf W \leq \sup W < \infty$, or

(b) for every $i \in N$ the σ-field $\mathcal{F}_i$ is finite,

then there exists $a > 0$ such that $\sum_{i \in N} X_i(\omega) \geq a$ at almost every $\omega \in \Omega$.

4 Information Advantage and Effort

In this section we study the impact of information advantage on the efforts that players exert in Tullock contests. It turns out that the impact of change in the players' information on effort is ambiguous, expect in restricted classes of contests. Throughout this section we restrict attention to the class of Tullock contests in which the players' cost of effort is state-independent, i.e., $W = 1_\Omega$.

The following lemma provides a system of equations that characterizes the equilibria of Tullock contests in this class. This system will be useful to derive properties of the equilibria of these contests. The proof of Lemma 1 is given in the Appendix.

Lemma 1. *Assume that $X = (X_1, ..., X_n)$ is an equilibrium of a Tullock contest. Then for all $i \in N$,*

$$E\left[\frac{X_i \bar{X}_{-i}}{\left(X_i + \bar{X}_{-i}\right)^2} V \mid \mathcal{F}_i\right] = X_i c'(X_i),$$

where $\bar{X}_{-i} = \sum_{j \in N \setminus \{i\}} X_j$.

Einy *et al.* (2017a) study the impact of public information on payoffs and effort in Tullock contests with symmetric information. Their Theorem 2.1 establishes that these contests have a unique equilibrium, which is symmetric and interior. For classic Tullock contests (where $c(x) = x$), they show that changes in the information available to the players have no impact on the expected effort they exert (Proposition 5.1). Using Lemma 1 we explicitly calculate the equilibrium efforts in these contests, which leads to the interesting observation that each individual's expected effort and total expected effort are both independent of the players' information, and that, while each individual's expected effort decreases with the number of players, the total effort increases with the number of players.

Proposition 2. *A classic Tullock contests in which players have symmetric information, i.e.,* $\mathcal{F}_i = \mathcal{G}$ *for all* $i \in N$*, where* $\mathcal{G}$ *is any* σ*-subfield of* $\mathcal{F}$*, has a unique equilibrium, which is symmetric and is given by* $(X, ..., X)$*, where* $X = (n-1)\, E\left[V \mid \mathcal{G}\right]/n^2$*. Hence the expected effort of each player,* $E[X] = (n-1)\, E\left[V\right]/n^2$*, and the expected total effort,* $E[nX] = (n-1)\, E\left[V\right]/n$*, are independent of* $\mathcal{G}$*, and while* $E[X]$ *decreases with the number of players,* $E[nX]$ *increases with the number of players.*

Proof. Assume that $\mathcal{F}_i = \mathcal{G}$ for all $i \in N$, and denote by X the player's strategy in the unique and symmetric equilibrium (see Theorem 2.1 of Einy et al. (2016)). Since $c'(x) \equiv 1$, Lemma 1 implies

$$X = E\left[\frac{(n-1)\, X^2}{(X+(n-1)X)^2} V \mid \mathcal{G}\right] = \frac{n-1}{n^2} E\left[V \mid \mathcal{G}\right].$$

Thus

$$E[nX] = nE[X] = n\left(\frac{n-1}{n^2} E[E[V \mid \mathcal{G}]]\right) = \frac{n-1}{n} E[V]. \blacksquare$$

Proposition 3 below shows that, in any equilibrium of a two-player Tullock contest in which the players' cost of effort is a convex function of the family $c(x) = x^{\alpha}$, the players' expected costs of effort coincide. (Example 1 in what follows will make clear that Proposition 3 does not extend to contests with more than two players.) Note that Proposition 3 does not involve any assumption about the players' information endowments; in particular, it holds when one player has information advantage over the other, but such a condition is not necessary.

Proposition 3. *Consider a two-player Tullock contest in which the players' cost of effort is* $c(x) = x^{\alpha}$*, where* $\alpha \in [1, \infty)$*. Then, in any equilibrium* (X_1, X_2)*,*

$$E[c(X_1)] = E[c(X_2)].$$

Proof. Let (X_1, X_2) be a equilibrium. Since $xc'(x) = x(\alpha x^{\alpha-1}) = \alpha c(x)$, Lemma 1 and the

Law of Iterated Expectation (see Theorem 34.3 in Billingsley (1995)) imply

$$\begin{aligned} E[c(X_i)] &= \frac{1}{\alpha}E[X_i c'(X_i)] \\ &= \frac{1}{\alpha}E\left[E\left[\frac{X_i X_j}{(X_i+X_j)^2}V \mid \mathcal{F}_i\right]\right] \\ &= \frac{1}{\alpha}E\left[\frac{X_i X_j}{(X_i+X_j)^2}V\right] \\ &= \frac{1}{\alpha}E\left[E\left[\frac{X_j X_i}{(X_j+X_i)^2}V \mid \mathcal{F}_j\right]\right] \\ &= \frac{1}{\alpha}E\left[X_j c'(X_j)\right] \\ &= E[c(X_j)]. \blacksquare \end{aligned}$$

Our next remark states an obvious but interesting implication of Proposition 3 for two-player *classic* Tullock contests. It turns out that in any equilibrium of such contests both players exert the same expected effort regardless of their information.

Remark 3. *In any equilibrium* (X_1, X_2) *of a two-player classic Tullock contest,* $E[X_1] = E[X_2]$.

Proposition 4 below establishes that when the convex cost function belongs to the family $c(x) = x^\alpha$ the expected effort of a player with information advantage is less than or equal to that of his opponent. This result is an implication of Jensen's inequality. Moreover, it is easy to see that the expected effort of the player with information advantage is strictly smaller than that of his opponent when $\alpha > 1$, except in equilibria in which the strategies of both players coincide almost everywhere.

Proposition 4. *Consider a two-player Tullock contest in which the players' cost of effort is* $c(x) = x^\alpha$*, where* $\alpha \in [1, \infty)$*, and in which player 2 has an information advantage over player 1. Then, in any equilibrium* (X_1, X_2)*,*

$$E[X_1] \geq E[X_2].$$

Proof. Let (X_1, X_2) be an equilibrium. Since $xc'(x) = \alpha c(x) = \alpha x^\alpha$, Lemma 1 and the assumption that $\mathcal{F}_1 \subset \mathcal{F}_2$ imply the following, using the law of iterated expectation:

$$X_1^\alpha = E\left[\frac{X_1 X_2 V}{\alpha (X_1+X_2)^2} \mid \mathcal{F}_1\right] = E\left[E\left[\frac{X_2 X_1 V}{\alpha (X_1+X_2)^2} \mid \mathcal{F}_2\right] \mid \mathcal{F}_1\right] = E\left[X_2^\alpha \mid \mathcal{F}_1\right].$$

Thus, using Jensen's inequality, the law of iterated expectation and $\mathcal{F}_1$-measurability of X_2, we obtain

$$\begin{aligned} E[X_1] &= E[(X_1^\alpha)^{\frac{1}{\alpha}}] \\ &= E[(E[X_2^\alpha \mid \mathcal{F}_1])^{\frac{1}{\alpha}}] \\ &\geq E[(E[X_2 \mid \mathcal{F}_1]^\alpha)^{\frac{1}{\alpha}}] \\ &= E[E[X_2 \mid \mathcal{F}_1]] \\ &= E[X_2]. \blacksquare \end{aligned}$$

Example 1 identifies a three-player classic Tullock contest with a unique equilibrium, in which a player with an information disadvantage exerts less effort than his opponents. Hence, neither of propositions 3, 4 nor remark 3 extend to Tullock contests with more than two players.

Example 1 *Consider a three-player classic Tullock contest in which* $\Omega = \{\omega_1, \omega_2\}$ *and* $p(\omega_1) = 1/8$*, and the value is* $V(\omega_1) = 1$ *and* $V(\omega_2) = 8$*. Assume that players* 2 *and* 3 *observe the value prior to taking action, but player 1 has only the prior information. The unique equilibrium of this contest is* X *given by* $(X_1(\omega_1), X_1(\omega_2)) = (168/121, 168/121)$ *and* $(X_2(\omega_1), X_2(\omega_2)) = (X_3(\omega_1), X_3(\omega_2)) = (0, 224/121)$*. Hence*

$$E[X_1] = \frac{168}{121} < \frac{7}{8} \cdot \frac{224}{121} = E[X_2] = E[X_3],$$

i.e., the expected effort of player 1 is less than those of players 2 and 3. (One can construct an example with this feature in which the equilibrium is interior, but the calculations involved are more cumbersome.)

Our next proposition shows that in two-player classic Tullock contests players exert, in expectation, less effort (and hence capture a larger share of the surplus) when one of them has an information advantage compared to the scenario when they are symmetrically informed.

Proposition 5. *In any interior equilibrium* (X_1, X_2) *of a two-player classic Tullock contest in which player 2 has information advantage over player 1, the expected total effort* $E[X_1] + E[X_2]$ *never exceeds* $E[V]/2$*, that is the expected total effort in a symmetric information scenario.*

Proof. Let (X_1, X_2) be a an interior equilibrium, i.e., $X_i > 0$ for $i \in \{1, 2\}$. By Lemma 1

$$E\left[\frac{X_1 X_2 V}{(X_1 + X_2)^2} \mid \mathcal{F}_2\right] = X_2.$$

Since both X_1 and X_2 are $\mathcal{F}_2$-measurable (as $\mathcal{F}_1 \subset \mathcal{F}_2$) and $X_2 > 0$, this equation may be written as

$$1 = E\left[\frac{X_1 V}{(X_1 + X_2)^2} \mid \mathcal{F}_2\right] = \frac{X_1 E\left[V \mid \mathcal{F}_2\right]}{(X_1 + X_2)^2},$$

i.e.,

$$X_2 = \sqrt{X_1}\sqrt{E[V \mid \mathcal{F}_2]} - X_1. \tag{6}$$

Also,

$$E\left[\frac{X_1 X_2 V}{(X_1 + X_2)^2} \mid \mathcal{F}_1\right] = X_1$$

by Lemma 1, and since $X_1 > 0$ is $\mathcal{F}_1$-measurable, we may write this equation as

$$E\left[\frac{X_2 V}{(X_1 + X_2)^2} \mid \mathcal{F}_1\right] = 1.$$

By the law of iterated expectation

$$E\left[\frac{X_2 V}{(X_1 + X_2)^2} \mid \mathcal{F}_1\right] = E\left[E\left[\frac{X_2 V}{(X_1 + X_2)^2} \mid \mathcal{F}_2\right] \mid \mathcal{F}_1\right].$$

Substituting X_2 from equation (6) and recalling that X_1 is $\mathcal{F}_2$-measurable, we get

$$\begin{aligned}
1 &= E\left[E\left[\frac{\left(\sqrt{X_1}\sqrt{E[V \mid \mathcal{F}_2]} - X_1\right) V}{\left(X_1 + \left(\sqrt{X_1}\sqrt{E[V \mid \mathcal{F}_2]} - X_1\right)\right)^2} \mid \mathcal{F}_2\right] \mid \mathcal{F}_1\right] \\
&= E\left[E\left[\frac{V}{\sqrt{X_1}\sqrt{E[V \mid \mathcal{F}_2]}} - \frac{V}{E[V \mid \mathcal{F}_2]} \mid \mathcal{F}_2\right] \mid \mathcal{F}_1\right] \\
&= \frac{1}{\sqrt{X_1}} E\left[\frac{E[V \mid \mathcal{F}_2]}{\sqrt{E[V \mid \mathcal{F}_2]}} \mid \mathcal{F}_1\right] - E\left[\frac{E[V \mid \mathcal{F}_2]}{E[V \mid \mathcal{F}_2]} \mid \mathcal{F}_1\right] \\
&= \frac{E\left[\sqrt{E[V \mid \mathcal{F}_2]} \mid \mathcal{F}_1\right]}{\sqrt{X_1}} - 1.
\end{aligned}$$

Hence

$$\sqrt{X_1} = \frac{E\left[\sqrt{E\left[V \mid \mathcal{F}_2\right]} \mid \mathcal{F}_1\right]}{2}, \tag{7}$$

Therefore, since $\mathcal{F}_2$ is finer than $\mathcal{F}_1$, Jensen's inequality implies

$$E[X_1] = \frac{E\left[\left(E\left[\sqrt{E[V \mid \mathcal{F}_2]} \mid \mathcal{F}_1\right]\right)^2\right]}{4} \leq \frac{E[V]}{4}.$$

Since $E(X_2) = E(X_1)$ by Remark 2,

$$E[X_1] + E[X_2] \leq \frac{E[V]}{2}.$$

By Proposition 2, $\frac{E[V]}{2}$ is precisely the expected total effort of the two players in a symmetric information scenario. ■

The following remark is an interesting observation implied by the equations (6) and (7) derived in the proof of Proposition 5.

Remark 4. *Consider a two-player classic Tullock contest in which player 2 observes the state of nature and player 1 has only the prior information, i.e.,* $\mathcal{F}_2 = \mathcal{F}$ *and* $\mathcal{F}_1 = \{\Omega, \varnothing\}$. *If the constest has an interior equilibrium, then it is given by* $X_1 = (E[\sqrt{V}])^2/4$, *and* $X_2 = E[\sqrt{V}]\left(V/2 - E[\sqrt{V}]/4\right)$.

Our last result establishes that in a two-player classic Tullock contest a player with information advantage wins the prize less often than his opponent, i.e., his ex-ante probability to win the prize is less or equal to $\frac{1}{2}$.

Proposition 6. *In any interior equilibrium* (X_1, X_2) *of a two-player classic Tullock contest in which player 2 has information advantage over player 1,*

$$E\left[\rho_2(X_1, X_2)\right] \leq \frac{1}{2} \leq E\left[\rho_1(X_1, X_2)\right].$$

Proof. Let (X_1, X_2) be a an interior equilibrium, i.e., $X_i > 0$ for $i \in \{1, 2\}$. Using the equations (6) and (7) derived in the proof of Proposition 5 we may write

$$\begin{aligned}
E\left[\rho_1(X_1, X_2)\right] &= E\left[\frac{X_1}{X_1 + X_2}\right] \\
&= E\left[\frac{X_1}{X_1 + \sqrt{X_1}\sqrt{E[V \mid \mathcal{F}_2] - X_1}}\right] \\
&= E\left[\frac{\sqrt{X_1}}{\sqrt{E[V \mid \mathcal{F}_2]}}\right] \\
&= \frac{1}{2} E\left[\frac{E\left[\sqrt{E\left[V \mid \mathcal{F}_2\right]} \mid \mathcal{F}_1\right]}{\sqrt{E[V \mid \mathcal{F}_2]}}\right] \\
&\geq \frac{1}{2},
\end{aligned}$$

where the last inequality follows from Jensen's inequality. ■

Our last example exhibits an eight-player classic Tullock contest in which a player who has an information advantage over the other players wins the prize more frequently, which shows that Proposition 6 does not extend to contests with more than two players.

Example 2 *Consider an eight-player classic Tullock contest in which* $\Omega = \{\omega_1, \omega_2\}$ *and* $p(\omega_1) = 1/2$. *The value is* $V(\omega_1) = 1$ *and* $V(\omega_2) = 2$. *Player 8 observes the value prior to taking action, while other players have only the prior information. The unique equilibrium of this contest* X *is given by* $X_1 = ... = X_7 = (x, x)$ *and* $X_8 = (0, y)$, *where*

$$x = \frac{7\sqrt{229} + 139}{1575}, \ y = \frac{56\sqrt{229} - 238}{1575}.$$

Thus, the ex-ante probability that player $i \in \{1, 2, ..., 7\}$ *wins the prize is*

$$\frac{1}{2} \cdot (\frac{1}{7} + \frac{x}{7x + y}) = \frac{\sqrt{229} + 37}{420},$$

whereas the ex-ante probability that player 8 *win the prize is*

$$1 - 7 \cdot \left(\frac{\sqrt{229} + 37}{420} \right) = \frac{161 - 7\sqrt{229}}{420} > \frac{\sqrt{229} + 37}{420},$$

i.e., the player with information advantage wins the prize more frequently than his opponents.

5 Tullock Contests and All-Pay Auctions

Consider a two-player contest in which $\Omega = \{\omega_1, \omega_2\}$ and $p(\omega_2) = p \in (0, 1)$. Player 2 observes the value prior to taking action, but player 1 has only the prior information. The value is $V(\omega_1) = 1$ and $V(\omega_2) = v \in (1, \infty)$. The cost has no random component: $W(\omega_1) = W(\omega_2) = 1$, and $c(x) = x$.

Assume that the prize is allocated using a Tullock contest. If $v < (1 + p)^2 / p^2$, then the unique equilibrium is interior, and is given by

$$X_1^{TC} = (x^2, x^2), \ X_2^{TC} = \left(x\,(1 - x)\,, x\left(\sqrt{v} - x\right)\right),$$

where $x = E[\sqrt{V}]/2 = [1 + p(\sqrt{v} - 1)]/2$. Hence the expected total effort is

$$TE^{TC} := E[X_1^{TC}] + E[X_2^{TC}] = [1 + p(\sqrt{v} - 1)]^2/2,$$

Likewise, if $v \geq (1+p)^2/p^2$, then the unique equilibrium is

$$\hat{X}_1^{TC} = (\hat{x}^2, \hat{x}^2),\ \hat{X}_2^{TC} = (0, \hat{x}(\sqrt{v} - \hat{x})),$$

where $\hat{x} = p\sqrt{v}/(1+p)$, and the expected total effort is

$$\widehat{TE}^{TC} := E[\hat{X}_1^{TC}] + E[\hat{X}_2^{TC}] = 2\hat{x} = 2p^2v/(1+p)^2.$$

Assume now that the prize is allocated using an all-pay auction. Using the formula provided by Einy *et al.* (2015b) we compute the players' total expected effort in the unique equilibrium, which is given by

$$TE^{APC} = 2(1-p)p + (1-p)^2 + p^2v.$$

Thus,

$$TE^{APA} - TE^{TC} = 2(1-p)p + \frac{1}{2}\left(1 - p - p\sqrt{v}\right)^2 > 0,$$

i.e., the all-pay auction generates more effort than the Tullock contest when $v < (1+p)^2/p^2$. However, if $v \geq (1+p)^2/p^2$, then

$$TE^{APA} - \widehat{TE}^{TC} = (1-p)(1+p) - p^2v\left(\frac{2}{(1+p)^2} - 1\right),$$

which may be negative – e.g., $p = 1/4$ and $v > 375/7$. Therefore, in general the level of effort generated by these two contests cannot be ranked.

6 Appendix

Proof of Remark 2.

Case (a). Assume w.l.o.g. that player 2 has an information advantage over player 1. Let $\varepsilon > 0$ be such that $c(3\varepsilon) < \frac{\inf V}{4 \sup W}$; it exists as $c(0) = 0$ and c is continuous at 0. Also, let $a \in (0, \varepsilon)$ be such that $\frac{2a}{\varepsilon + 2a} < \frac{\left[c(\varepsilon) - c(\frac{\varepsilon}{2})\right] \cdot \inf W}{\sup V}$; it exists because the left-hand side vanishes when $a \searrow 0$, while the right-hand side is positive. Now consider an equilibrium X in the contest. We will show that $X_1 \geq a$ a.e. on Ω. Assume by the way of contradiction that this is false. Then there exists a positive-measure set $A_1 \in \mathcal{F}_1$ such that $X_1 < a$ on A_1. We will now show that $X_2 \leq \varepsilon$ a.e. on A_1.

Indeed, suppose to the contrary that $X_2 > \varepsilon$ on some positive-measure $A_2 \in \mathcal{F}_2$ which is a subset of A^1. Consider a strategy $X_2' = \frac{\varepsilon}{2} \cdot 1_{A_2} + X_i \cdot 1_{\Omega \setminus A_2} \in S_2$. Then, by switching from X_2 to X_2', player 2 decreases his expected reward by at most $\frac{2a}{\varepsilon+2a}\sup V \cdot p(A_2)$, and simultaneously decreases his expected cost by at least $\left[c(\varepsilon) - c(\frac{\varepsilon}{2})\right] \cdot \inf W \cdot p(A_2)$. By the choice of a the first expression is smaller than the second, and hence deviating to X_2' is, in expectation, profitable for player 2, in contradiction to X being an equilibrium.

It follows that $\max\{X_1(\omega), X_2(\omega)\} \leq \varepsilon$ on A_1. Let i be a player for whom $E(\rho_i(X) \mid A_1) \leq \frac{1}{2}$, and consider a strategy $X_i'' = 3\varepsilon \cdot 1_{A_1} + X_i \cdot 1_{\Omega \setminus A_1} \in S_i$. Since $A_1 \in \mathcal{F}_1 \subset \mathcal{F}_2$, X_i'' is measurable w.r.t. both $\mathcal{F}_1$ and $\mathcal{F}_2$. Also $\rho_i(X) \leq \rho_i(X_{-i}, X_i'')$ on A_1, and $E(\rho_i(X_{-i}, X_i'') \mid A_1) \geq \frac{3}{4}$ (this is due to the fact that, on A_1, $\rho_i(X_{-i}, X_i'') \geq \frac{3\varepsilon}{3\varepsilon+\varepsilon} = \frac{3}{4}$). Thus, by switching from X_i to X_i'' player i improves his expected reward by at least $\frac{1}{4}\inf V p(A_1)$, while incurring an expected cost increase of at most $c(3\varepsilon) \cdot \sup W \cdot p(A_1)$. By the choice of ε, such a deviation leads to a net gain in the expected utility, in contradiction to X being an equilibrium. We conclude that, indeed, $X_1(\omega) \geq a$ for a.e. $\omega \in \Omega$.

Case (b). As $\sum_{i \in N} X_i$ is measurable w.r.t. $\vee_{i \in N}\mathcal{F}_i$ – the smallest σ-field containing each $\mathcal{F}_i$ (which is, in particular, finite), the probabilities $p\left(\sum_{i \in N} X_i \geq a\right)$ can take only finitely many values in $[0,1]$. Let $\delta = \max_{a>0} p\left(\sum_{i \in N} X_i \geq a\right)$, and suppose that it is attained at $a_0 > 0$. By Remark 1, $\sum_{i \in N} X_i > 0$ a.e. on Ω in any equilibrium X, and hence $\lim_{a \searrow 0} p\left(\sum_{i \in N} X_i \geq a\right) = p\left(\sum_{i \in N} X_i > 0\right) = 1$. Therefore $\delta = 1$ and a_0 is the desired bound for the equilibrium sum of efforts. ■

Proof of Lemma 1. Let X be an equilibrium and let $i \in N$. For any $\varepsilon \in \mathbb{R}$ set $X_{i,\varepsilon}' = \max\{X_i + \varepsilon, 0\} \in S_i$. The equilibrium condition (3) implies

$$E[u_i(\cdot, X(\cdot)) \mid \mathcal{F}_i] \geq E[u_i(\cdot, X_{-i}(\cdot), X_{i,\varepsilon}'(\cdot)) \mid \mathcal{F}_i].$$

In particular, for any $\varepsilon > 0$

$$E\left[\frac{u_i(\cdot, X_{-i}(\cdot), X_{i,\varepsilon}'(\cdot)) - u_i(\cdot, X(\cdot))}{\varepsilon} \mid \mathcal{F}_i\right] \leq 0, \tag{8}$$

and hence

$$E\left[\frac{u_i(\cdot, X_{-i}(\cdot), X_{i,-\varepsilon}'(\cdot)) - u_i(\cdot, X(\cdot))}{-\varepsilon} \mid \mathcal{F}_i\right] \geq 0. \tag{9}$$

As X_i and $X_{i,-\varepsilon}'$ are $\mathcal{F}_i$-measurable and non-negative, by multiplying both sides of inequality (8) by X_i we obtain

$$E\left[X_i(\cdot) \frac{u_i(\cdot, X_{-i}(\cdot), X_{i,\varepsilon}'(\cdot)) - u_i(\cdot, X(\cdot))}{\varepsilon} \mid \mathcal{F}_i\right] \leq 0. \tag{10}$$

Likewise, by multiplying both sides of the inequality (9) by $X'_{i,-\varepsilon}$ we obtain

$$E\left[X'_{i,-\varepsilon}\left(\cdot\right)\frac{u_i(\cdot,X_{-i}\left(\cdot\right),X'_{i,-\varepsilon}\left(\cdot\right))-u_i(\cdot,X\left(\cdot\right))}{-\varepsilon}\mid\mathcal{F}_i\right]\geq 0. \tag{11}$$

For every $\omega\in\Omega$ the function $u_i(\omega,x)$ is concave in the variable x_i, and hence for any $\varepsilon>0$

$$\begin{aligned}&\left|X_i\left(\omega\right)\frac{u_i(\omega,X_{-i}\left(\omega\right),X'_{i,\varepsilon}\left(\omega\right))-u_i(\omega,X\left(\omega\right))}{\varepsilon}\right| \\ \leq\ & X_i\left(\omega\right)\max\left\{\left|\frac{d}{dx_i}u_i(\omega,X\left(\omega\right))\right|,\left|\frac{d}{dx_i}u_i(\omega,X_{-i}\left(\omega\right),X_{i,\varepsilon}\left(\omega\right))\right|\right\}\end{aligned} \tag{12}$$

and

$$\begin{aligned}&\left|X'_{i,-\varepsilon}\left(\omega\right)\frac{u_i(\omega,X_{-i}\left(\omega\right),X'_{i,-\varepsilon}\left(\omega\right))-u_i(\omega,X\left(\omega\right))}{-\varepsilon}\right| \\ \leq\ & X'_{i,-\varepsilon}\left(\omega\right)\max\{\left|\frac{d}{dx_i}u_i(\omega,X\left(\omega\right))\right|,\left|\frac{d}{dx_i}u_i(\omega,X_{-i}\left(\omega\right),X_{i,-\varepsilon}\left(\omega\right))\right|\}\end{aligned} \tag{13}$$

(When $\sum_{j=1}^{n}x_j=0$ the partial derivatives $du_i\left(\omega,x\right)/dx_i$ may not be defined. However, the bounds in (12) and (13) will vanish in this case, and are thus well-defined.)

Since the cost function c is convex and strictly increasing, there exists $b>0$ such that $c(b)>E(V)$. It follows that X_i is bounded from above by b almost everywhere on Ω, as otherwise the expected equilibrium payoff of player i would be negative, making the deviation $\hat{X}_i\equiv 0$ profitable. Now notice that

$$\frac{d}{dx_i}u_i(\omega,X(\omega))=\frac{\bar{X}_{-i}(\omega)}{\left(X_i(\omega)+\bar{X}_{-i}(\omega)\right)^2}V\left(\omega\right)-c'(X_i(\omega)) \tag{14}$$

whenever $X_i(\omega)+\bar{X}_{-i}(\omega)>0$. Since X_i is bounded as argued above, (14) implies that the functions

$$\begin{aligned}&X_i\left(\cdot\right)\frac{d}{dx_i}u_i(\cdot,X\left(\cdot\right)),\\&X_i\left(\cdot\right)\frac{d}{dx_i}u_i(\cdot,X_{-i}\left(\cdot\right),X_{i,\varepsilon}\left(\cdot\right)),\\&X'_{i,-\varepsilon}\left(\cdot\right)\frac{d}{dx_i}u_i(\cdot,X\left(\cdot\right)),\\&X'_{i,-\varepsilon}\left(\cdot\right)\frac{d}{dx_i}u_i(\cdot,X_{-i}\left(\cdot\right),X_{i,-\varepsilon}\left(\cdot\right))\end{aligned}$$

are bounded; in particular, the expressions in both (12) and (13) are uniformly bounded. Since for every $\omega \in \Omega$

$$\begin{aligned} & \lim_{\varepsilon \searrow 0} X_i(\omega) \frac{u_i(\omega, X_{-i}(\omega), X'_{i,\varepsilon}(\omega)) - u_i(\omega, X(\omega))}{\varepsilon} \\ = & \lim_{\varepsilon \searrow 0} X'_{i,-\varepsilon}(\omega) \frac{u_i(\omega, X_{-i}(\omega), X'_{i,-\varepsilon}(\omega)) - u_i(\omega, X(\omega))}{-\varepsilon} \\ = & X_i(\omega) \frac{d}{dx_i} u_i(\omega, X(\omega)), \end{aligned}$$

the bounded convergence theorem implies that the convergence as $\varepsilon \searrow 0$ is also in the L_1-norm on $(\Omega, \mathcal{F}, p)$, which in turn implies pointwise a.e. convergence of the conditional expectation on the σ-field $\mathcal{F}_i$. Thus, the functions in (10) and (11) converge a.e. to

$$E\left[X_i(\cdot) \frac{d}{dx_i} u_i(\cdot, X(\cdot)) \mid \mathcal{F}_i\right],$$

and hence we obtain that

$$E\left[X_i(\cdot) \frac{d}{dx_i} u_i(\cdot, X(\cdot)) \mid \mathcal{F}_i\right] = 0$$

a.e. on Ω. Using (14) we may write this equation as

$$\begin{aligned} 0 &= E\left[X_i(\cdot) \frac{d}{dx_i} u_i(\cdot, X(\cdot)) \mid \mathcal{F}_i\right] \\ &= E\left[\frac{X_i \overline{X}_{-i}}{\left(X_i + \overline{X}_{-i}\right)^2} V - X_i c'(X_i) \mid \mathcal{F}_i\right] \\ &= E\left[\frac{X_i \overline{X}_{-i}}{\left(X_i + \overline{X}_{-i}\right)^2} V \mid \mathcal{F}_i\right] - E\left[X_i c'(X_i) \mid \mathcal{F}_i\right] \\ &= E\left[\frac{X_i \overline{X}_{-i}}{\left(X_i + \overline{X}_{-i}\right)^2} V \mid \mathcal{F}_i\right] - X_i c'(X_i), \end{aligned}$$

which establishes the lemma. ■

Proof that equation (5) used in the proof of Proposition 1. We rely on this proof on the notations and intermediate results derived in the proof Lemma 1. Note that the proof of Lemma 1 does not involve Proposition 1.

For every $\omega \in \Omega$

$$\begin{aligned} & \lim_{\varepsilon \to 0+} 1_{X_i > 0}(\omega) \times \frac{u_i(\omega, X_{-i}(\omega), X'_{i,\varepsilon}(\omega)) - u_i(\omega, X(\omega))}{\varepsilon} \\ = & \lim_{\varepsilon \to 0+} 1_{X_i > 0}(\omega) \times \frac{u_i(\omega, X_{-i}(\omega), X'_{i,-\varepsilon}(\omega)) - u_i(\omega, X(\omega))}{-\varepsilon} \\ = & 1_{X_i > 0}(\omega) \times \frac{d}{dx_i} u_i(\omega, X(\omega)). \end{aligned} \tag{15}$$

Also, for every $\omega \in \Omega$ the function $u_i(\omega, x)$ is concave in the variable x_i, and hence for any $\varepsilon \in \left(0, \frac{a}{2}\right)$ and $\omega \in \Omega_i$

$$\begin{aligned} & \left| \frac{u_i(\omega, X_{-i}(\omega), X'_{i,\varepsilon}(\omega)) - u_i(\omega, X(\omega))}{\varepsilon} \right| \\ \leq \quad & \max\{\left| \frac{d}{dx_i} u_i(\omega, X(\omega)) \right|, \left| \frac{d}{dx_i} u_i(\omega, X_{-i}(\omega), X_{i,\varepsilon}(\omega)) \right|\} \end{aligned} \tag{16}$$

and

$$\begin{aligned} & \left| \frac{u_i(\omega, X_{-i}(\omega), X'_{i,-\varepsilon}(\omega)) - u_i(\omega, X(\omega))}{-\varepsilon} \right| \\ \leq \quad & \max\{\left| \frac{d}{dx_i} u_i(\omega, X(\omega)) \right|, \left| \frac{d}{dx_i} u_i(\omega, X_{-i}(\omega), X_{i,-\varepsilon}(\omega)) \right|\}. \end{aligned} \tag{17}$$

Since X_i is bounded (as was shown in the proof of Lemma 1), and $\sum_{i \in N} X_i \geq a$ a.e. on Ω by assumption (implying in particular that $\bar{X}_{-i} + X_{i,-\varepsilon} \geq \frac{a}{2}$ for $\varepsilon \in \left(0, \frac{a}{2}\right)$), it follows from (14) that the right-hand side functions in both (16) and (17) are bounded from above by the same integrable function $f = \frac{X_{i,\frac{a}{2}}}{a^2} V + c'(X_{i,\frac{a}{2}})$. Using this fact, (15), and the conditional dominated convergence theorem (see Corollary 3.1.1. (iv) in Borkar (1995)), we obtain that, almost surely on Ω,

$$\begin{aligned} & \lim_{\varepsilon \to 0+} E\left[1_{X_i > 0}(\cdot) \times \frac{u_i(\cdot, X_{-i}(\cdot), X'_{i,\varepsilon}(\cdot)) - u_i(\cdot, X(\cdot))}{\varepsilon} \mid \mathcal{F}_i \right] \\ = \quad & \lim_{\varepsilon \to 0+} E\left[1_{X_i > 0}(\cdot) \times \frac{u_i(\cdot, X_{-i}(\cdot), X'_{i,-\varepsilon}(\cdot)) - u_i(\cdot, X(\cdot))}{-\varepsilon} \mid \mathcal{F}_i \right] \\ = \quad & E\left[1_{X_i > 0}(\cdot) \times \frac{d}{dx_i} u_i(\cdot, X(\cdot)) \mid \mathcal{F}_i \right]. \end{aligned} \tag{18}$$

As $1_{X_i > 0}$ is $\mathcal{F}_i$-measurable and can be extracted from the expectation, multiplying by $1_{X_i > 0}$ both sides of the inequalities (8) and (9), and using (18), we obtain

$$E\left[1_{X_i > 0}(\cdot) \times \frac{d}{dx_i} u_i(\cdot, X(\cdot)) \mid \mathcal{F}_i \right] = 0,$$

which is equation (5). ■

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