Midterm – March 2015

Exercise 1. In an economy that extends over two periods, today and tomorrow, there are two consumers, A and B, and a single perishable good, consumption. The state of nature tomorrow can either be H or L. Both consumers are endowed with twelve units of the good in each period, and their preferences over consumption today and tomorrow are represented by the utility functions $u^A(c_0^A, c_H^A, c_L^A) = \ln c_0^A + 2\ln c_H^A$ and $u^B(c_0^B, c_H^B, c_L^B) = \ln c_0^B + 2\ln c_L^B$, respectively.

(a) (15 points) Suppose that the only market is a credit market. Calculate how much will each person $i \in \{A, B\}$ borrow or save, b_i, as a function of the interest rate, r. Calculate the competitive equilibrium interest rate and allocation. Is this allocation Pareto optimal?

(b) (15 points) Now suppose that in addition to the credit market there is a market for a security that pays return (one unit of the good) only in state H. Calculate how much will each person borrow or save and how many units of the security will buy or sell as a function of the interest rate and the security price, p. Calculate the competitive equilibrium interest rate and security price, as well as the allocation. Is this allocation Pareto optimal?

(a) Consumer i's budget constraint is

$$\begin{aligned} c_0^i &\leq 12 + b^i \\ c_H^i &\leq 12 - (1+r)b^i \\ c_L^i &\leq 12 - (1+r)b^i, \end{aligned}$$

where b is the number of units she borrows today, and r is the interest rate. Since u^A is increasing in c_0^A and c_H^A, then $c_0^A = 12 - b^i$ and $c_H^A = 12 + (1+r)b^A$. Likewise, since u^B is increasing in c_0^B and c_L^B, then $c_0^B = 12 + b^i$ and $c_L^B = 12 - (1+r)b^B$. Hence consumer $i \in \{A, B\}$ chooses the amount she borrows in order to solve

$$\max_{b \in R} \ \ln\left(12 + b^i\right) + 2\ln\left(12 - (1+r)\, b^i\right)$$

Thus, both consumers solve the same problem, and therefore their solutions coincide, i.e., $b^A(r) = b^B(r)$. Hence in a competitive equilibrium there is not borrowing or lending, i.e., $b^A(r) + b^B(r) = 0$ only if $b^A(r) = b^B(r)$. Hence the competitive allocation is just the initial endowment,

$$(c_0^A, c_H^A, c_L^A) = (c_0^B, c_H^B, c_L^B) = (12, 12, 12)$$

Obviously, this allocation is not Pareto optimal. For example, the allocation

$$(c_0^A, c_H^A, c_L^A) = (12, 24, 0), \ (c_0^B, c_H^B, c_L^B) = (12, 0, 24)$$

is Pareto superior.

(b) With credit and security markets consumer i's budget constraint is

$$\begin{aligned}
c_0^i &\leq 12 + b^i - py^i \\
c_H^i &\leq 12 - (1+r)b^i + y \\
c_L^i &\leq 12 - (1+r)b^i,
\end{aligned}$$

where y^i is the number of units of the security consumer i buys.

Consumer B only cares about her consumption today and tomorrow in state L, c_0^B and c_L^B. By selling security and borrowing ($y^B, b^B < 0$), she can increase her consumption tomorrow in state L. Of course, she has to balance her budget in state H, i.e.,

$$12 - (1+r)b^B + y^B \geq 0. \tag{1}$$

If $1 + r \geq 1/p$, then by setting $py^B = b^B = -M$, she maintains her consumption today at $c_0^B = 12$, increases her consumption tomorrow in state L to $c_L^B = 12-(1+r)(-M)$, and is able balance her budget in state H since $12-(1+r)(-M)+(-M/p) = 12+(1+r-1/p)M > 0$. Thus, when $1 + r \geq 1/p$, consumer A will want to sell an infinite amount of security and borrow an infinite amount. Therefore in a competitive equilibrium the interest rate is negative. Let us assume henceforth that $(1 + r)\,p < 1$.

Consumer B sets up y^B to solve equation (1), i.e., $y^B = -12 + (1 + r)b^B$, and chooses her demand of credit b^B to solve

$$\max_{b^B \in \mathbb{R}} \ \ln\left(12 + b^B - \left(-12 + (1+r)b^B\right)p\right) + 2\ln\left(12 - (1+r)\,b^B\right).$$

An interior solution to this problem satisfies the first order condition

$$\frac{1-(1+r)p}{24 - rb^B} - \frac{2\,(1+r)}{12-(1+r)\,b^B} = 0,$$

Hence

$$b^B(r,p) = \frac{12(3+p)(1+r) + 12r}{(r+1)\,(p(1+r)+2r-1)}.$$

and

$$\begin{aligned}
y^B(r,p) &= -12 + (1+r)b^B(r,p) \\
&= 12\left(\frac{(3+p)(1+r)+r}{p(1+r)+2r-1} - 1\right).
\end{aligned}$$

Likewise, consumer A by buying security financed with credit is able to increase her consumption in state H, e.g., by choosing $y^A = b^A = M > 0$ is able to maintain her consumption today at $c_0^A = 12$ and increase her consumption tomorrow in state H to $c_H^B = 12 - rM$. (recall that $r < 0$). Since the utility of consumer A is strictly increasing in c_H^A, she would like to exploit this possibility as much as possible, but is constrained as she has to payback her credit when the state is L, that is, she is limited by the constraint

$$0 \leq 12 - (1+r)b^A.$$

Therefore, consumer A will borrow as much as possible, that is,

$$b^A(r,p) = \frac{12}{1+r},$$

and choose y^A to solve

$$\max_{y^A \in \mathbb{R}} \ \ln\left(12 + \frac{12}{1+r} - py^A\right) + 2\ln\left(12 - (1+r)\frac{12}{1+r} + y^A\right).$$

An interior solution to this problem satisfies the first order condition

$$\frac{-p}{12 + \frac{12}{1+r} - y^A} + \frac{2}{y^A} = 0.$$

Solving for y^A we get

$$y^A(r,p) = \frac{24(2+r)}{(r+1)(p+2)}$$

Solving the market clearing conditions

$$\begin{aligned} b^A(r,p) + b^B(r,p) &= 0 \\ y^A(r,p) + y^B(r,p) &= 0. \end{aligned}$$

we obtain $(r^*, p^*) = (-1/2, 1)$, and $b^A(r^*, p^*) = -b^B(r^*, p^*) = 24$, $y^A(r^*, p^*) = -y^B(r^*) = 24$. The competitive allocation is therefore

$$(c_0^A, c_H^A, c_L^A) = (12, 24, 0), \ (c_0^B, c_H^B, c_L^B) = (12, 0, 24),$$

which is Pareto optimal.

Exercise 2. A coastal city is considering building a small artificial beach. It is known that some residents value this public project in $\bar{v} = 2$ monetary units, whereas other residents have no value for it (that is, $\underline{v} = 0$). In order to make a decision, the city council is going to conduct a survey asking each resident whether his/her value is $\bar{v}$. If a majority of residents answer "yes," then the beach will be built and its cost will be shared equally by those who answered "yes" – that is, those who answered "no" will be exempted from paying.

(a) (5 points) Describe the Bayesian game faced by the residents of the city. Assume that the residents values are independent realizations of a random variable that takes values are $\bar{v}$ and $\underline{v}$ with probabilities $q \in (0,1)$ and $1-q$, respectively.

(b) (15 points) Assume that there are 3 residents, and that the cost of the project is 3 monetary units. Determine the set of values of q for which sincere voting is a Bayesian Nash equilibrium. Is the sincere BNE efficient?

(c) (10 points) For the values of q for which sincere voting is not a BNE, is there a (mixed strategy) symmetric BNE? In this BNE efficient?

(a) The Bayesian game played by the residents, $\Gamma = (N, T, A, u, p)$, is described by the following ingredients:

- $N = \{1, ..., n\}$
- $T = \{\bar{v}, \underline{v}\}^n$
- $A = \{0,1\}^n$, *where* 0 *(respectively* 1*) represents answering no (yes)*
- $u_i : A \times T \to R$ *is defined as*

$$u_i(a,t) = \begin{cases} 0 & \text{if } \sum_{j=1}^n a_j < n/2 \\ 0 & \text{if } \sum_{j=1}^n a_j \geq n/2,\ t_i = \underline{v} \text{ and } a_1 = 0 \\ -3/\sum_{j=1}^n a_j & \text{if } \sum_{j=1}^n a_j \geq n/2,\ t_i = \underline{v} \text{ and } a_1 = 1 \\ 2 & \text{if } \sum_{j=1}^n a_j \geq n/2,\ t_i = \bar{v} \text{ and } a_1 = 0 \\ 2 - 3/\sum_{j=1}^n a_j & \text{if } \sum_{j=1}^n a_j \geq n/2,\ t_i = \bar{v} \text{ and } a_1 = 1. \end{cases}$$

- *For each* $t \in T$, $p(t) = \prod_{j=1}^n \rho(t_i)$, *where* $\rho(\bar{v}) = q$ *and* $\rho(\underline{v}) = 1 - q$.

(b) The sincere strategy is $s_i^*(\underline{v}) = 0$ and $s_i^*(\bar{v}) = 1$.

An inspection of the function u_i reveals that when the individual's value is $\underline{v}$ answering "yes" is a (weakly) dominated action: in the best case it leads to a payoff equal to zero, and in the worst case it leads to a negative payoff. Therefore answering "no" (that is, saying the truth, $s_i^*(\underline{v}) = 0$) is optimal when $t_i = \underline{v}$.

The expected payoff of a resident i whose value is $t_i = \bar{v}$ and answers (sincerely) "yes" ($a_i = 1$), when the other residents answer sincerely as well is

$$U_i(s_{-i}^*, 1/\bar{v}) = q^2\,(2-1) + 2q\,(1-q)\left(2 - \frac{3}{2}\right) = q,$$

whereas her expected utility when she lies (that is, answer "no", $a_i = 0$) is

$$U_i(s_{-i}^*, 0/\bar{v}) = 2q^2.$$

In order for sincere behavior to be a BNE we need $q \geq 2q^2$, that is, $q \leq 1/2$.

The sincere BNE are efficient because the beach is built only when its social value ($t_1 + t_2 + t_3$) is greater than its cost (3), since only in this case there are at least two individual whose value is $\bar{v}$ and therefore respond "yes."

(c) Assume that $q > 1/2$. Since answering "yes" ($a_i = 1$) is (weakly) dominated the individual's value is $\underline{v}$, let us consider the mixed strategy profile σ^* given for each $i \in N$, by $\sigma_i^*(1/\underline{v}) = 0$ and $\sigma_i^*(1/\bar{v}) = \alpha \in (0,1)$. We have

$$U_i(\sigma_{-i}^*, 1/\bar{v}) = q^2\alpha^2\,(2-1) + q^2\,(2\alpha\,(1-\alpha))\left(2 - \frac{3}{2}\right) + 2q\,(1-q)\,\alpha\left(2 - \frac{3}{2}\right) = q\alpha,$$

and

$$U_i(\sigma_{-i}^*, 0/\bar{v}) = 2q^2\alpha^2.$$

In order for σ^* to be a BNE we need that $U_i(\sigma_{-i}^*, 1/\bar{v}) = U_i(\sigma_{-i}^*, 0/\bar{v})$, that is,

$$q\alpha = 2q^2\alpha^2.$$

Hence $\alpha^* = 1/2q$. Since $q > 1/2$, we have $\alpha^* < 1$. Therefore when sincere behavior is not a BNE there is a symmetric BNE in mixed strategies.

These BNE in mixed strategies are inefficient since there is a positive probability that beach is not built despite the fact that its social value is greater than its cost.

In addition, there is a symmetric pure strategy BNE $\hat{s}$ in which the three residents respond always "no", i.e., $\hat{s}_i(\underline{v}) = \hat{s}_i(\bar{v}) = 0$ for all $i \in N$, since when the residents follow $\hat{s}$ the beach is not built, and if a single player changes his answer the outcome does not change, i.e. $U_i(\hat{s}) = U_i(\hat{s}_{-i}, s_i) = 0$ for all $i \in N$. This is a somewhat trivial equilibrium.

Exercise 3. Consider the contract design problem of a Principal whose revenue is a random variable taking values $x_1 = 4$ and $x_2 = 18$ with probabilities that depend on whether or not the Agent exerts effort, $e \in \{0, 1\}$; specifically, $p_1(0) = p_2(0) = 1/2$, whereas $p_1(1) = 1/4$ and $p_2(1) = 3/4$. The Agent's reservation utility is $\underline{u} = 1$, and his cost of effort is $v(1) = 1 > 0 = v(0)$.

(a) (20 points) Assuming that the Principal is risk-neutral and the preferences of the Agent are represented by the von Neumann-Morgenstern utility function $u(x) = \sqrt{x}$, determine the optimal contract when effort is verifiable.

(b) (10 points) Under the assumptions in (a), determine the optimal contract when effort is *not* verifiable.

(c) (10 points) Assuming that the Principal is risk averse and the Agent is risk-neutral, determine the optimal contract when effort is verifiable and when it is *not* verifiable.

(a) The optimal contract not requiring the Agent exerting effort, i.e., $e = 0$*, involves paying the Agent a fixed wage* $\bar{w}(0) = 1$*, which is obtained solving the participation constraint with equality,*

$$Eu(\bar{w}(0)) = \underline{u} + v(0) \Leftrightarrow \sqrt{\bar{w}(0)} = 1 + 0.$$

Profit is $y(0) = E(X(0) - 1) = 10$.

The optimal contract requiring the Agent exerting effort (i.e., $e = 1$*) involves paying the Agent a fixed wage* $\bar{w}(1) = 4$*, which is obtained solving the participation constraint with equality,*

$$Eu(\bar{w}(1)) = \underline{u} + v(1) \Leftrightarrow \sqrt{\bar{w}(1)} = 1 + 1.$$

Profit is $y(1) = E\left(X(1)\right) - 4) = 10.5$.

Hence when effort is verifiable the optimal contract is $(e^*, \bar{w}(e^*)) = (1, 4)$.

(b) If effort is not verifiable, then the contract $(0, \bar{w}(0))$ *continues to satisfy the participation and incentive constraints. However, if the Principal wants the Agent to exert effort, then the wage contract* $W^*(1) = (w_1(1), w_2(1))$ *must satisfy the participation and incentive constraint with equality, that is,*

$$\begin{aligned} \frac{1}{4}\sqrt{w_1(1)} + \frac{3}{4}\sqrt{w_2(1)} &= 2 \\ \frac{1}{4}\sqrt{w_1(1)} + \frac{3}{4}\sqrt{w_2(1)} - 1 &= \frac{1}{2}\sqrt{w_1(1)} + \frac{1}{2}\sqrt{w_2(1)} \end{aligned}$$

Unfortunately, this system does not have a real solution. (I am sorry, I did not plan this – it is just a mistake.)

If we assume that the minimum wage the Principal can pay is $w = 0$*, then the most favorable contract is* $w_1(1) = 0$ *and*

$$\begin{aligned} \frac{3}{4}\sqrt{w_2(1)} &\geq 2 \\ \frac{1}{4}\sqrt{w_2(1)} &\geq 1, \end{aligned}$$

that is, $w_2(1) = 16$*. The Principal's profit with this contract is*

$$E\left(X(1)\right) - E(W^*(1)) = \left(\frac{1}{4}\right)(4) + \left(\frac{3}{4}\right)(18 - 16) = \frac{5}{2} < 10.$$

Hence the optimal contract is $(e^*; W(e^*)) = (0; 1, 1)$.

(c) In this case the optimal contract involves a franchise, i.e., a fixed payment to the Principal y^* *given by*

$$y^* = \max_{e \in \{0,1\}} E(X(e)) - v(e) - \underline{u}.$$

If we normalize the Agent's utility function to be $u(x) = x$*, and maintain his reservation utility at the level* $\underline{u} = 1$*, then our calculations in part (a) yield* $y^* = 12.5$*, corresponding to optimal level of effort* $e^* = 1$.