Final Exam (May 2020)

Exercise 1. An exchange economy populated by two individuals operates over two periods, today and tomorrow. There is uncertainty of whether tomorrow will be cold or hot. There is a single perishable consumption good, and the preferences over consumption today (x), tomorrow if cold (y), and tomorrow if hot (z) of individuals 1 and 2 are represented by the utility functions $u_1(x, y, z) = x(2y + z)$ and $u_2(x, y, z) = x(y + 3z)$, while their endowments are $(\bar{x}_1, \bar{y}_1, \bar{z}_1) = (\bar{x}_2, \bar{y}_2, \bar{z}_2) = (10, 10, 10)$. There are no contingent markets, but there is a credit market and a market for a security that pays 1 unit of consumption tomorrow if cold and nothing if hot. Determine the competitive equilibrium interest rate r^*, security price q^*, and allocation.

(Hint. Normalize the spot prices to 1 in every date and state. Note that $MRS^1_{yz}(x, y, z) = 2 > 1/3 = MRS^2_{yz}(x, y, z)$. Guess that in the CE the *effective price ratio* p_y/p_z is some intermediate value in the interval $(1/3, 2)$, and derive the consequences over the consumption of y and z by individuals 1 and 2. Use these results and the budget constrains to simplify the problem of calculating how much consumer i's borrows $b_i(q, r)$ and how many units of the security she demands $s_i(q, r)$. Then set up the market clearing conditions and check that $q^* = 1/2$ and $r^* = 0$ solve this system. Finally, calculate how much each consumer borrows or lends and how many units of the security buys or sells for (q^*, r^*), and use these calculations and again the budget constrains to calculate the CE allocation.)

Solution: For (r, q), the problem of consumer i is

$$\begin{array}{l} \max_{[(x,y,z),(b,s)]\in\mathbb{R}^3_+\times\mathbb{R}^2} u_i(x, y, z) \\ \text{subject to:} \\ \quad x \leq 10 + b - qs \\ \quad y \leq 10 - (1 + r)b + s \\ \quad z \leq 10 - (1 + r)b. \end{array}$$

Since u_i is increasing in x, y, and z, then budget constraints are binding at the solution. Moreover, if the CE values of q and r are such that the effective price of y relative to that of z is in the interval $(1/3, 2)$, as suggested, then $z^*_1 = y^*_2 = 0$. Using these values and solving for b and s in the budget constraints, we see that

$$b_1(q, r) = \frac{10}{1 + r}$$

and

$$s_2(r, q) = (1 + r)b_2(r, q) - 10.$$

Solving for x_1 and z_1 as consumer 1's utility as a function of s (her demand of security), we may rewrite her problem as

$$\max_{s\in\mathbb{R}}(10 - qs + \frac{10}{1 + r})\,(2s)\,.$$

Taking derivative and solving the first order condition for an interior solution we get

$$s_1^* = s_1(q,r) = \frac{5(2+r)}{q(1+r)}.$$

Likewise, we may write consumer 2's utility as a function of b (her demand of credit) and rewrite her problem as

$$\max_{b\in\mathbb{R}} \left(10(1+q) + (1-q(1+r))b\right)\left(3\left(10-(1+r)b\right)\right).$$

Taking derivative and solving the first order condition for an interior solution we get

$$b_2(q,r) = \frac{10q(1+r)+5r}{(1+r)\left(q(1+r)-1\right)},$$

and hence

$$s_2(q,r) = \frac{10q(1+r)+5r}{q(1+r)-1} - 10.$$

The CE equilibrium interest rate and security price, (r^*, q^*), solve the system

$$\begin{aligned} \frac{10}{1+r} + \frac{10q(1+r)+5r}{(1+r)\left(q(1+r)-1\right)} &= 0 \\ \frac{5(2+r)}{q(1+r)} + \frac{10q(1+r)+5r}{q(1+r)-1} - 10 &= 0 \end{aligned}$$

which yields

$$(q^*, r^*) = \left(\frac{1}{2}, 0\right).$$

Substituting, we get

$$b_1(q^*, r^*) = -b_2(q^*, r^*) = 10,$$

and

$$s_1(q^*, r^*) = -s_2(q^*, r^*) = 20.$$

Hence the equilibrium allocation is

$$\begin{aligned} (x_1^*, y_1^*, z_1^*) &= (10 - q^* s_1(q^*, r^*) + b_1(q^*, r^*), 10 - (1+r^*)b_1(q^*, r^*) + s_1(q^*, r^*), 10 - (1+r^*)b_1(q^*, r^*)) \\ &= (10 - \frac{1}{2}(20) + 10, 10 - (1)10 + 20, 10 - (1)10) \\ &= (10, 20, 0) \end{aligned}$$

and

$$\begin{aligned} (x_2^*, y_2^*, z_2^*) &= (10 - q^* s_2(q^*, r^*) + b_2(q^*, r^*), 10 - (1+r^*)b_2(q^*, r^*) + s_2(q^*, r^*), 10 - (1+r^*)b_2(q^*, r^*)) \\ &= (10 - \frac{1}{2}(-20) + (-10), 10 - (1)(-10) + (-20), 10 - (1)(-10)) \\ &= (10, 0, 20). \end{aligned}$$

Exercise 2. A village has a common grazing land to support the cows owned by its 100 inhabitants, each of whom is allowed to own at most one cow. A cow yield of milk is $f(x) = 120 - x$ quarts, where x is the total number of cows grazing in the common land.

(a) Calculate the per-capita consumption of milk assuming that the inhabitants of the village decide independently and simultaneously whether or not to own one cow. (Assume the objective of each inhabitant is to maximize his/her milk consumption.)

(b) Calculate the number of cows that maximizes the total yield of milk.

(c) Assume now that the village's council charges a fee (Lindahl price) of p_L quarts of milk to the inhabitants who choose to own a cow, and distributes the revenue equally among the inhabitants who choose not to own a cow. Determine the Lindahl price that maximizes the per-capita milk consumption and calculate the milk consumption of the inhabitants who choose to own cow, and that of those who choose not to own a cow. (Hint. For which value of p_L the Nash equilibrium of this game induces the inhabitants to own the number of cows that maximizes the total yield of milk?)

Solution. Since at most 100 people may own a cow, an individual owning a cow gets at least

$$(120 - 100)1 = 20$$

quarts of milk. Since the cost of owning a cow is zero, every individual will own a cow, i.e., $z_i^{VC} = 1$*. Hence the number of cows in the grazing land will be*

$$z^{VC} = \sum_{i=1}^{10} z_i^{VC} = 100,$$

and the per capita consumption of milk will be

$$m^{VC} = (12 - 10)1 = 20.$$

The number of cows grazing in the common land that maximizes milk production is identified by solving the problem

$$\max_{z \in \{0,1,\ldots,10\}} M(Z) = (120 - Z)\, Z.$$

We have

$$M'(Z) = (120 - 2Z) = 0 \Leftrightarrow Z^* = 60.$$

With this number of cows the total yield of milk is

$$M^* = (120 - 60)60 = 3600 \textit{ quarts.}$$

With the Lindahl scheme, the payoffs of an inhabitant who chooses to own a cow when $n - 1$ *other inhabitants choose to own a cow is* $a(n) = (120 - n) - p_L$*, whereas her payoff is* $b(n) = (n-1)p_L/(100-(n-1))$ *if she chooses not to own a cow. The Lindahl price* p_L *that maximizes the total yield of milk is such that in equilibrium exactly* 6 *inhabitants choose to own a cow. For this* *to be an equilibrium, we must have* $a(60) \geq b(60)$ *and* $b(60) \geq a(60)$; *that is, when 5 inhabitants own a cow and 4 other do not own a cow, the remaining inhabitant is indifferent between owning a cow or not. Thus,* p_L *must satisfy*

$$60 - p_L = \frac{59p_L}{41}$$

i.e.,

$$p_L = \frac{123}{5} = 24.6.$$

Note that $a(n)$ *decreases with* n *and* $b(n)$ *increases with* n. *Hence, with the Lindahl price,* $p_L = 24.6$, *if* $n > 60$, *then an inhabitant owning a cow can increase its payoff by choosing not to own one, while if* $n < 60$, *then an inhabitant not owning a cow can increase its payoff by choosing to own one. Therefore a Nash equilibrium of the game* $n^* = 60$. *Interestingly, the milk consumption of the inhabitants who owns a cow is* $a(60) = 60 - 24.6 = 35.4$ *quarts, whereas the milk consumption of the inhabitants who do not own a cow is* $b(40) = 60\,(24.6)\,/40 = 36.9$ *quarts.*

Exercise 3. The revenue of a risk-neutral principal is a random variable X taking values $x_1 = 4$, $x_2 = 8$ and $x_3 = 12$ with probabilities t $p_1(e) = p_2(e) = (1-e)/2$ and $p_3(e) = e$, respectively, where e is the level of effort of an agent. Agents may exert any effort in the interval $[0, 1]$. There are two types of agents L and H with identical preferences represented by the Bernoulli utility function $u(w) = \sqrt{w}$, and identical reservation utility $\underline{u} = 0$, but their costs of effort differ, and are given for $e \in [0, 1]$ by $c_L(e) = 2e$ and $c_H(e) = 3e$

(a) Assume that an agent's type is observable and effort is verifiable. Determine the contract the Principal will offer to each type of agent and the Principal's profit in either case. recall that maximal effort is optimal for the low cost type since the Principal's profit strictly increasing for $e \in [0, 1]$ for the agent type.

(b) Assume that an agent's type is observable, but *effort is not verifiable.* Determine the Principal's optimal contract to the type L Agent, that is, the optimal random wage $W^* = (w_1^*, w_2^*, w_3^*)$, and effort e^*. . Also, calculate the cost of moral hazard to the Principal. (Note that since the revenue realizations x_1 and x_2 are equally likely regardless of the effort, you may directly assume that the optimal random wage satisfies $w_1^* = w_2^* = w_{12}^*$.)

(c) Now assume that effort is verifiable, but the Principal *does not observe an agent's type.* Agents of type H and L are present in the population of agents in equal fractions. (In the notation used in class, this means $q = 1/2$.) Identify the Principal's optimal menu of contracts for each value of q.

Solution.

(a) *The Principal offers the contract* $(\bar{w}_\tau, \bar{e}_\tau)$ *to the type* $\tau \in \{H, L\}$ *agent, where* $\bar{w}_\tau = w_\tau(\bar{e}_\tau)$ *and* $w_\tau(e)$ *is given by the participation constraint,*

$$\sqrt{w_L(e)} = 2e + \underline{u}, \text{ and } \sqrt{w_H(e)} = 3e + \underline{u}.$$

that is, $w_L(e) = 4e_L^2$ *, and* $w_H(e) = 9e_H^2$*, and* $\bar{e}_\tau$ *solves the problem*

$$\max_{e \in [0,1]} \pi_\tau(e) = \mathbb{E}[X(e)] - w_\tau(e).$$

Since

$$\mathbb{E}[X(e)] = \frac{1-e}{2}(4+8) + 12e = 6 + 6e,$$

then

$$\pi_L(e) = 6e + 6 - 4e^2,$$

Hence $\pi_L'(e) = 6 - 8e = 0$*, implies* $\bar{e}_L = 3/4$ *and* $\bar{w}_L = 4\bar{e}_L^2 = 9/4$*, and therefore*

$$\pi_L(\bar{e}_L) = \mathbb{E}[X(\frac{3}{4})] - \frac{9}{4} = 6\left(\frac{3}{4}\right) + 6 - \frac{9}{4} = \frac{33}{4}.$$

Also,

$$\pi_H(e) = 6e + 6 - 9e^2,$$

and therefore the first order condition for a solution to the Principal's problem is

$$6 - 18e = 0,$$

i.e., $\bar{e}_H = 1/3$ *and* $\bar{w}_H = 9\bar{e}_L^2 = 1$, *and therefore*

$$\pi_H(\bar{e}_H) = \mathbb{E}[X(\frac{1}{3})] - 1 = 7.$$

(b) Given a wage offer W, Agent's payoff as a function of his effort is

$$\mathbb{E}[u(W(e))] - c_L(e) = (1-e)\sqrt{w_{12}} + e\sqrt{w_3} - 2e = \sqrt{w_{12}} + (\sqrt{w_3} - \sqrt{w_{12}} - 2)\,e,$$

which is linear in effort. Hence in order for the Agent to be willing to exert effort we must have

$$\sqrt{w_3} - \sqrt{w_{12}} - 2 \geq 0.$$

Also, for the Agent to accept the contract, the participation constrain must hold, that is,

$$\mathbb{E}[u(W(e))] \geq c_L(e).$$

Since these two constraints are binding at the solution, we solve for the wage offer as a function of effort. However, the solution to the system formed by the two constraints is independent of effort, and is given by $W^* = (0,0,4)$. (Note that the Principal must set up a wage equal to zero for revenue realization other than x_3 in order to provide appropriate incentives.) Hence $\mathbb{E}[W^*] = 4e$, and the optimal effort maximizes the Principal's profit,

$$\mathbb{E}[X(e)] - \mathbb{E}[W^*] = 6 + 6e - 4e = 6 + 2e.$$

Since expected profit is increasing in effort, maximum effort is optimal, i.e., the optimal contract involving a positive effort is $[W^*, 1]$. The Principal's expected profit with this contract is

$$\mathbb{E}[X(1)] - \mathbb{E}[W^*] = 8.$$

Alternatively, the Principal may offer a contract involving no effort, $[\tilde{W}, \tilde{e}] = [(0,0,0), 0]$, which satisfies both the participation and incentives constraints. This contract generates the profit

$$\mathbb{E}[X(0)] - \mathbb{E}[\tilde{W}] = 6 - 0 = 6.$$

This contract is dominated by the contract $[W^*, 1]$, which is therefore optimal.

The cost of moral hazard imposes to the Principal is therefore

$$\pi_L(\bar{e}_L) - 8 = \frac{33}{4} - 8 = \frac{1}{4}.$$

Since expected profit is increasing in effort, maximum effort is optimal. Note that the expected wage paid by the Principal is 4, greater than that of part (a), although effort requested is also greater. Also nota the the Principal's expected profit with the contract is

$$\mathbb{E}[X(1)] - \mathbb{E}[W^*(1)] = 8.$$

The cost of moral hazard imposes to the Principal is therefore $33/4 - 8 = 1/4$.

(c) The Principal may offer the single contract $(w, e) = (3/4, 9/4)$*, which only type L agents accept, leading to an expected profit of*

$$\Pi_H = \frac{1}{2}\left(\mathbb{E}\left[X(3/4)\right] - 9/4\right) = \frac{33}{8}$$

Alternatively, she may design an incentive compatible menu of contracts. As shown in class, such menu must satisfy the participation constraint of the type H,

$$\sqrt{w_H} = 3e_H, \tag{1}$$

and the incentive of the type L,

$$\sqrt{w_L} - 2e_L = \sqrt{w_H} - 2e_H. \tag{2}$$

Also, the menu must satisfy the optimality equations (derived in class),

$$\begin{aligned}
\left(\mathbb{E}\left[X(e_H)\right]\right)' &= \frac{kc'(e_H)}{u'(w_H)} + \frac{1-q}{q}(k-1)\frac{c'(e_H)}{u'(w_L)} \\
\left(\mathbb{E}\left[X(e_L)\right]\right)' &= \frac{c_L'(e_H)}{u'(w_L)}
\end{aligned}$$

which in this exercise become

$$\begin{aligned}
6 &= \frac{\frac{3}{2}(3)}{\frac{1}{2\sqrt{w_H}}} + (1)\left(\frac{3}{2} - 1\right)\frac{3}{\frac{1}{2\sqrt{w_L}}} = 9\sqrt{w_H} + 3\sqrt{w_L} &(3) \\
6 &= \frac{2}{\frac{1}{2\sqrt{w_L}}} = 4\sqrt{w_L}. &(4)
\end{aligned}$$

The solution to this system of equations yields the menu

$$[(w_L, e_L), (w_H, e_H)] = [(\frac{9}{4}, \frac{13}{18}), (\frac{1}{36}, \frac{1}{18})]$$

The Principal's expected profit with this menu is

$$\begin{aligned}
\frac{1}{2}\left(\mathbb{E}\left[X(\frac{13}{18})\right] - \frac{9}{4}\right) + \frac{1}{2}\left(\mathbb{E}\left[X\left(\frac{1}{18}\right)\right] - \frac{1}{36}\right) &= \frac{1}{2}\left(6 + 6\left(\frac{13}{18}\right) - \frac{9}{4}\right) + \frac{1}{2}\left(6 + 6\left(\frac{1}{18}\right) - \frac{1}{36}\right) \\
&= \frac{259}{36} \simeq 7.2 > \frac{33}{8}.
\end{aligned}$$

Hence, this menu is optimal. Obviously the Principal's profit is less than her profit with complete information,

$$\frac{1}{2}\left(\frac{33}{4}\right) + \frac{1}{2}(7) = \frac{61}{8} = 7.625 > \frac{259}{36},$$

so that adverse selection imposes a cost to the Principal.

Since effort of the high and low cost agents is lower than under complete information, so the adverse selection has a social cost. Finally, the low cost agent captures the rents

$$w_L - c_L(13/18) = \frac{9}{4} - 2\left(\frac{13}{18}\right) = \frac{29}{36}.$$