Final Exam 2019

Exercise 1. An economy operates over two dates, today and tomorrow. The state of the economy tomorrow is uncertain: it can be in a boom (B) or in a recession (R). There is a single perishable good, consumption, and two consumers with preferences for consumption today (x), consumption tomorrow if B (y), and consumption tomorrow if R (z) represented by the utility function $u_i(x, y, z) = x + 4\sqrt{y} + 8\sqrt{a_i z}$, where $a_1 = 1$ and $a_2 = 2$, and endowments $(\bar{x}_1, \bar{y}_1, \bar{z}_1) = (\bar{x}_2, \bar{y}_2, \bar{z}_2) = (8, 9, 6)$.

(a) (10 points) Identify the set of interior Pareto optimal allocations.

(b) (30 points) Assume that there are only two markets in which agents trade today a risky security r that pays 1 unit of consumption tomorrow if the economy is in a boom and nothing if it is in a recession, and a safe security s that pays 1 unit of consumption tomorrow regardless of the state of the economy. Identify the CE allocation and security prices. (Hint: It may help to calculate the Arrow-Debreu CE equilibrium prices, $(1, p_y^*, p_z^*)$, and then relate these prices to the security prices, (q_r^*, q_s^*), in the Radner CE. Note that securities are traded today, and their prices are given in units of consumption today. Alternatively, you may try to calculate the consumers' demands of securities, $r_i(q_r, q_s)$ and $s_i(q_r, q_s)$, and solve for the prices that clear these markets, $(q_r^*, q_s^*) = (2/3, 8/3)$. Then calculate the consumers security trades and the allocation of consumption goods they imply. Whatever your approach, you may assume without proof that the CE equilibrium is unique and interior.)

Exercise 2. Consider the contract design problem of a risk-neutral Principal who wants to hire an Agent. The Principal's revenue is a random variable $X(e)$ that takes the value $x_1 = 4$ with probability $p(e) = (1 - e)/4$, and $x_2 = 16$ with probability $1 - p(e)$, where $e \in [0, 1/2]$ is the Agent's effort. The Agent's preferences are represented by the Bernoulli utility function $u(w) = \sqrt{w}$, and his reservation utility is $\underline{u} = 1$. An agent's cost of effort is $c(e) = e$ if he is of the low cost type L, and it is $2c(e)$ if he is of the high cost type H.

(a) (10 points) Identify the optimal contracts for high and low cost agents when effort is verifiable and the Agent's type is observable. Calculate the Principal's expected profits.

(b) (10 points) Identify the optimal contracts when the only feasible levels of effort are $e \in \{0, 1/2\}$, and effort is NOT verifiable, but agent's types are observable.

(c) (10 points) Identify the optimal menu of contracts when effort is verifiable, but the Agent's type is NOT observable, assuming that both types are present in equal numbers in the population of agents. (Hint: If you solve the system of equations identifying the interior solution to the Principal's problem when designing this menu, you are going to notice that it involves values for effort that are not feasible – that is, an interior solution does not exist. You me want to reconsider the menu you identified in part (a) and think about its properties.)

Exercise 3. A coastal town is building an artificial beach, which involves buying and transporting sand from neighboring islands at a cost of 200 monetary units per hundred meters of beach. The preferences of each of their 100 residents are described by a function of the form $u(x,y) = y - (v - x)^2$, where x is the length of the beach (in hundreds of meters), and y is the resident's income. The value of v is $\bar{v} = 4$ for n the residents who are merchants (to whom the beach, which attract tourists, has a business value in addition to a private value), and it is $\underline{v} = 2$ for the remaining $100 - n$ residents.

(a) (15 points) Identify the Pareto optimal size of the beach to be built and the Lindahl prices as a function of n.

(b) (5 points) Determine the size of beach that will be built if the cost is to be covered by voluntary contributions.

(c) (10 points) Now assume that the city council makes the decision by asking each resident to declare her preferred size, x_i, and then build a beach of a size equal to the *median* of $(x_1, ..., x_{100})$, taxing residents equally to pay the cost. (The median M is the smallest number such that for at least 50 residents $x_i \leq M$.) Is sincere voting a dominant strategy? What beach size will be built?

Solutions

Exercise 1. (a) The set of interior Pareto optimal allocations is identified by the optimality conditions

$$\begin{aligned} MRS^1_{yx}(x_1, y_1, z_1) &= \frac{2}{\sqrt{y_1}} = \frac{2}{\sqrt{y_2}} = MRS^2_{yx}(x_2, y_2, z_2) \\ MRS^1_{zx}(x_1, y_1, z_1) &= \frac{4}{\sqrt{z_1}} = \frac{4\sqrt{2}}{\sqrt{z_2}} = MRS^2_{zx}(x_2, y_2, z_2), \end{aligned}$$

and the feasibility constraints

$$\begin{aligned} x_1 + x_2 &= \bar{x}_1 + \bar{x}_2 = 16 \\ y_1 + y_2 &= \bar{y}_1 + \bar{y}_2 = 18 \\ z_1 + z_2 &= \bar{z}_1 + \bar{z}_2 = 12. \end{aligned}$$

Thus, an interior Pareto optimal allocation satisfies $y_1 = y_2 = 9$, $z_1 = 4$, $z_2 = 8$, and $x_1 + x_2 = 12$.

(b) Since the returns matrix

$$R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is non-singular, this market structure is equivalent to a complete set of markets. Hence, the CE allocation is Pareto optimal and, if it is interior (which we assume for now), it satisfies $y_1^* = y_2^* = 9$, $z_1^* = 4$, $z_2^* = 8$. *Denote by* (p_x^*, p_y^*, p_z^*) *the CE Arrow-Debreu prices. Then*

$$\begin{aligned} \frac{p_y^*}{p_x^*} &= MRS^1_{yx}(x_1^*, y_1^*, z_1^*) = MRS^2_{yx}(x_2^*, y_2^*, z_2^*) = \frac{2}{3}, \\ \frac{p_z^*}{p_x^*} &= MRS^1_{zx}(x_1^*, y_1^*, z_1^*) = MRS^2_{zx}(x_2^*, y_2^*, z_2^*) = 2. \end{aligned}$$

If we normalize $p_x^* = 1$, *then* $p_y^* = 2/3$ *and* $p_z^* = 2$. *We can then calculate the consumption today of consumers 1 and 2 as*

$$x_i^* = \bar{x}_i + 2/3(\bar{y}_i - y_i^*) + 2(\bar{z}_i - z_i^*) = \bar{x}_i + 2(\bar{z}_i - z_i^*),$$

i.e.,

$$\begin{aligned} x_1^* &= 8 + 2\,(6 - 4) = 12 \\ x_2^* &= 8 + 2\,(6 - 8) = 4. \end{aligned}$$

Now, since q_r *is the (effective) price of a unit of* y *in units of* x, *in the CE the price of this security satisfies*

$$q_r^* = \frac{p_y^*}{p_x^*} = \frac{2}{3}.$$

And since $q_s - q_r$ *is the (effective) price of a unit of* z *in units of* x*, we must have*

$$q_s^* - q_r^* = \frac{p_z^*}{p_x^*} = 2 \Leftrightarrow q_s^* = \frac{8}{3}.$$

In order to arrive at the equilibrium allocation in the economy with security markets, the agents security trades must satisfy

$$\begin{aligned} y_i^* &= \bar{y}_i + r_i^* + s_i^* \\ z_i^* &= \bar{z}_i + s_i^*. \end{aligned}$$

Hence $r_1^* = -s_1^* = -r_2^* = s_2^* = 2$*. One can readily verify that at the equilibrium prices the equations*

$$x_i^* = \bar{x}_i - q_r^* r_i^* - q_s^* s_i^*$$

hold for $i \in \{1, 2\}$*.*

Alternatively, we can calculate the consumers' security demands and find the securities equilibrium prices by solving the market clearing conditions. For $i \in \{1, 2\}$ *budget constraints are*

$$\begin{aligned} x &\leq \bar{x}_i - q_r r - q_s s \\ y &\leq \bar{y}_i + r + s \\ z &\leq \bar{z}_i + s, \end{aligned}$$

which at the solution are binding since u_i *is increasing with respect to all goods.*

Substituting the value of initial endowments we can write consumer i*'s problem as*

$$\max_{(b,z) \in \mathbb{R}^2} v_i(r, s) = (8 - q_r r - q_s s) + 4\sqrt{9 + r + s} + 8a_i\sqrt{6 + s}$$

The first order conditions for a solution to this problem are

$$\begin{aligned} \frac{\partial v_i}{\partial r} &= -q_r + \frac{2}{\sqrt{9 + r + s}} = 0 \\ \frac{\partial v_i}{\partial s} &= -q_s + \frac{2}{\sqrt{9 + r + s}} + \frac{4a_i}{\sqrt{6 + s}} = 0. \end{aligned}$$

Solving the system we get

$$\begin{aligned} r_i(q_r, q_s) &= \frac{4}{q_r^2} - \frac{16a_i^2}{(q_s - q_r)^2} - 3 \\ s_i(q_r, q_s) &= \frac{16a_i^2}{(q_s - q_r)^2} - 6. \end{aligned}$$

Market clearing requires

$$\begin{aligned}\frac{8}{q_r^2} - \frac{16(1+2)}{(q_s - q_r)^2} - 6 &= 0 \\ \frac{16(1+2)}{(q_s - q_r)^2} - 12 &= 0.\end{aligned}$$

Substituting $(q_s - q_r)^2 = 4$ *(from the second equation) into the first equation we get* $q_r^2 = 4/9$. *Hence* $q_r^* = 2/3$, *and* $q_s^* = 2 + q_r^* = 8/3$.

Hence

$$\begin{aligned}r_1(q_r^*, q_s^*) &= \frac{4}{\left(\frac{2}{3}\right)^2} - \frac{16}{\left(\frac{8}{3} - \frac{2}{3}\right)^2} - 3 = 2 = -r_2(q_r^*, q_s^*) \\ s_1(q_r^*, q_s^*) &= \frac{16}{\left(\frac{8}{3} - \frac{2}{3}\right)^2} - 6 = -2 = -s_2(q_r^*, q_s^*).\end{aligned}$$

Using the budget constraints we calculate the resulting allocation as

$$\begin{aligned}x_1^* &= 8 - \frac{2}{3}(2) - \frac{8}{3}(-2) = 12,\ x_2^* = 8 - \frac{2}{3}(-2) - \frac{8}{3}(2) = 4 \\ y_1^* &= 9 + r_1 + s_1 = 9,\ y_2^* = 9 + r_2 + s_2 = 9 \\ z_1^* &= 6 + s_1 = 4,\ z_2^* = 6 + s_2 = 8.\end{aligned}$$

Exercise 2. (a) The expected revenue as a function of effort is

$$\mathbb{E}[X(e)] = \frac{1-e}{4}(4) + \left(1 - \frac{1-e}{4}\right) 16 = 13 + 3e.$$

For type $i \in \{H, L\}$ the optimal contract involves a fixed wage satisfying the participation constrain with equality, i.e.,

$$\sqrt{w_H} = 2e + \underline{u}, \text{ and } \sqrt{w_L} = e + \underline{u}.$$

Hence

$$\bar{w}_H(e) = (2e+1)^2, \text{ and } \bar{w}_L(e) = (e+1)^2.$$

In order to identify the optimal effort that each type of contract $i \in \{H, L\}$ should involve, we solve the problem

$$\max_{e \in [0,1]} \mathbb{E}[\pi_i(e)] = \mathbb{E}[X(e)] - \bar{w}_i(e).$$

Taking derivatives we get

$$\mathbb{E}'[\pi_H(e)] = \mathbb{E}'[X(e)] - \bar{w}'_H(e) = 3 - 4(2e+1) = -1 - 8e < 0,$$

which implies $e^*_H = 0$, and

$$\mathbb{E}'[\pi_L(e)] = \mathbb{E}'[X(e)] - \bar{w}'_L(e) = 3 - 2(e+1) = 0$$

which implies $e^*_L = 1/2$. Hence the optimal contracts are

$$(e^*_H, \bar{w}_H) = (0, 1), \ \ (e^*_L, \bar{w}_L) = (1/2, 9/4).$$

The Principal's expected profit are

$$\mathbb{E}[\pi^*_H] = \mathbb{E}[X(0)] - \bar{w}_H = 13 + 3(0) - 1 = 12,$$

and

$$\mathbb{E}[\pi^*_L] = \mathbb{E}[X(1/2)] - \bar{w}_L = 13 + 3\left(\frac{1}{2}\right) - \frac{9}{4} = 12.25.$$

(b) Since the cost of exerting no effort is zero for both types of agents, the contract $(e, \bar{w}) = (0, 1)$ is acceptable both types of agents, i.e., satisfies the participation and incentive constraints. The expected profits from these contracts is

$$E[X(0)] - \bar{w} = 12.$$

Since $p(1/2) = 1/8$ and $p(0) = 1/2$, an acceptable incentive compatible contract $W_L = (w_1, w_2)$ involving effort $e = 1/2$ by a type L agent must satisfy

$$\begin{aligned} \frac{\sqrt{w_1}}{8} + \frac{7\sqrt{w_2}}{8} &= \frac{1}{2} + 1 \\ \frac{\sqrt{w_1}}{8} + \frac{7\sqrt{w_2}}{8} - \frac{1}{2} &= \frac{\sqrt{w_1}}{2} + \frac{\sqrt{w_2}}{2}, \end{aligned}$$

which solution is $W_L = (1/9, 25/9)$*. Hence the Principal's expected profit from this contract is*

$$\mathbb{E}[X(1/2)] - \mathbb{E}[W_L(1/2)] = 9 + 3(1/2) - \left(\frac{1}{8}\left(\frac{1}{9}\right) + \frac{7}{8}\left(\frac{25}{9}\right)\right) = \frac{145}{18} \simeq 8.05.$$

This contract is not optimal, since the contract $(e, \bar{w}) = (0, 1)$ *generates greater profits.*

Likewise, a contract $W = (w_1, w_2)$ *inducing an agent of type* H *to exert effort* $e = 1/2$ *will require a contract involving an even higher expected wage, and will generate less profits that the contract* $(e, \bar{w}) = (0, 1)$*.*

Hence the optimal contract to offer to both types of agent is $(e, \bar{w}) = (0, 1)$ *leading to an expected profit equal to*

$$\mathbb{E}[\pi] = \mathbb{E}[X(0)] - \bar{w} = 12.$$

(c) The solution to the Principal's problem is easy to spot: as shown in part (a) in the absence of adverse selections the optimal is $\{(e_H^*, \bar{w}_H) = (0, 1), (e_L^*, \bar{w}_L) = (1/2, 9/4)\}$*. Notice that the contract offered the high cost type* H*,* $(e_H^*, \bar{w}_H) = (0, 1)$*, does not generate rents to the low cost type. (This is because it involves no effort, and the cost of exerting no effort is the same for both types.) Obviously, by design the contract offered the low cost type* L*,* $(e_L^*, \bar{w}_L) = (1/2, 9/4)$*, does not generate rents either. Moreover, this contract is not acceptable by the* H *type. Hence, this menu satisfies the incentive constraint of both, the high and the low cost type. Since it satisfies the participation constraints of both types, this menu is feasible in the presence of adverse selection. Hence, it is optimal in this case too.*

(If you try to find the optimal contracts as an interior solution to the Principal's problem, which is identify by system of equations

$$\begin{aligned}
(\mathbb{E}[X(e_H)])' &= \frac{kc'(e_H)}{u'(w_H)} + \frac{1-q}{q}(k-1)\frac{c'(e_H)}{u'(w_L)} \\
(\mathbb{E}[X(e_L)])' &= \frac{c'(e_L)}{u'(w_L)} \\
u(w_H) &= kc(e_H) + \underline{u} \\
u(w_L) - c(e_L) &= u(w_H) - c(e_H),
\end{aligned}$$

which for the primitives of this exercise is given by

$$\begin{aligned}
3 &= 4\sqrt{w_H} + 2\sqrt{w_L} \\
3 &= 2\sqrt{w_L} \\
\sqrt{w_H} &= 2e_H + 1 \\
\sqrt{w_L} - e_L &= \sqrt{w_H} - e_H.
\end{aligned}$$

you will find that the solution involves $e_L = 3/2$ *and* $e_H = -1/2$*. Hence an interior solution does not exist. Indeed, the solution identified above is a corner solution to this problem.)*

Exercise 3. (a) For each resident,

$$MRS(x,y) = 2\,(v - x)\,.$$

Hence the sum of the MRS of the residents are

$$2n\,(\bar{v} - x) + 2(100 - n)\,(\underline{v} - x) = 200\,(V(n) - x)\,,$$

where

$$V(n) = \underline{v} + \frac{n}{100}\,(\bar{v} - \underline{v}) = 2 + \frac{2n}{100}$$

is the average of the residents' ideal size for the beach. The optimal size of the beach is the solution to the equation

$$200\,(V(n) - x) = 200;$$

i.e.,

$$x^*(n) = V(n) - 1 = 1 + \frac{2n}{100}.$$

For example, if 25% of the town residents are merchants, the optimal size of the beach is $x^*(25) = 1.5$ (i.e., 150 meters.)

(b) The utility of a resident who contributes $z \geq 0$, when the other residents jointly contribute $\bar{Z} \geq 0$ is

$$v(z, \bar{Z}) = y - z - \left(v - \frac{z + \bar{Z}}{200}\right)^2.$$

Taking derivatives

$$\frac{\partial v}{\partial z} = -1 + \frac{2}{200}\left(v - \frac{z + \bar{Z}}{200}\right) \leq -1 + \frac{2}{200}\left(4 - \frac{0 + 0}{200}\right) = -1 + \frac{1}{25} < 0.$$

That is, a single resident is not willing to contribute anything to build the beach: simply, building a beach of even an infinitesimal size is too expensive for a single individual. Hence under voluntary contributions the beach is not built.

(c) The sincere strategy is to declare the size that solves the problem

$$\max_{x \geq 0} y - 2x - (v - x)^2$$

where $2x = 200x/100$ is the taxed paid by the resident if a beach of size x is built. The first order condition for a solution to this problem is

$$-2 + 2(v - x) = 0.$$

Thus, the beach size the maximizes the resident's utility is $x_i = v_i - 1$.

It is easy to see that sincere voting, i.e., declaring $x_i = v_i - 1$, is a dominant strategy: if $M = v_i - 1$, then declaring a different size cannot make the individual better off, and may make him worse off. If $M > v_i - 1$ (respectively, $M < v_i - 1$) the individual cannot decrease (increase) the size of the beach by changing the size she declares, and hence cannot benefit from a deviation from her ideal size.

If merchant and non-merchant residents were to pay $2p$ *and* p*, respectively, then* $2pn + (100 - n)p = 200$*, i.e.,* $p = 200/(100 + n)$*. Then residents will declare the solution to the problem*

$$\max_{x \geq 0} y - \alpha_i px - (v_i - x)^2 ,$$

where $\alpha_i = 2$ *for merchant residents and* $\alpha_i = 1$ *for non-merchant residents. Thus, a resident declares* x *solving the equation* $2v_1 - \alpha_i p = 2x$

$$-\alpha_i p + 2(v_i - x) = 0,$$

that is,

$$x_i = v_i - \frac{\alpha_i p}{2}.$$

In equilibrium, if $n > 50$ *a beach of size*

$$\underline{v} - p/2 = 2 - 100(100 + n)$$

is built, whereas is $n > 50$*, a beach of size*

$$M = M = \bar{v} - p = 4 - 200(100 + n)$$

is built.