Final Exam (May 30, 2018): Solve any three exercises.

Exercise 1. (40 points) In an economy that extends over two dates, today and tomorrow, there is a single perishable good, consumption. The state of nature tomorrow can be either sunny or cloudy. There are two consumers with preferences for consumption today (x), consumption tomorrow if sunny (y), and consumption tomorrow if cloudy (z), represented by the utility functions $u_1(x, y, z) = xy$, $u_2(x, y, z) = xz$, and initial endowments $(\bar{x}_1, \bar{y}_1, \bar{z}_1) = (4, 0, 2)$, and $(\bar{x}_2, \bar{y}_2, \bar{z}_2) = (0, 4, 2)$, respectively. There are spot markets for consumption each date (prices are normalized to one, i.e., $p_x = p_y = p_z = 1$). There are two other markets that operate today, a credit market in which the interest rate is $R = 1 + r$, and a market for a security θ (which price is q) that pays 2 units of good in the second date if sunny and nothing if cloudy. . Calculate a competitive equilibrium allocation, as well as the amount consumers borrow/lend and the number of units of the security they buy/sell. (Use the notation $b_i(R, q)$ and $\theta_i(R, q)$ for consumer i's demands of credit and security. You should arrive at the conclusion that the CE prices are $(R^*, q^*) = (1, 6/5)$.)

Consumer 1's problem is

$$\max_{(x,y,z),b,\theta \in \mathbb{R}^3_+ \times \mathbb{R} \times \mathbb{R}} xy,$$
$$s.t.\ x = 4 + b - q\theta,\ y = 2\theta - Rb,\ z = 2 - Rb.$$

Since he does not care about z, we may assume that in equilibrium he will lend as much as possible conditional on being able to payback if the state is cloudy, i.e., he sets $b_1(R, q) = 2/R$. Then we may write his problem as

$$\max_{\theta \in \mathbb{R}} (4 + \frac{2}{R} - q\theta)(2\theta - 2).$$

Solving the F.O.C. for a solution to this problem we get

$$\theta_1(R, q) = \frac{4R + Rq + 2}{2Rq}.$$

Likewise, consumer 2's problem is

$$\max_{(x,y,z),b,\theta \in \mathbb{R}^3_+ \times \mathbb{R} \times \mathbb{R}} xz,$$
$$s.t.\ x = b - q\theta,\ y = 4 + 2\theta - Rb,\ z = 2 - Rb.$$

Since Consumer 2 does not care about y, then for $q > 0$ he will sell as much asset as possible, that is, he will set $y = 4 + 2\theta - Rb = 0$, or, $\theta = Rb/2 - 2$, and choose b to solve

$$\max_{b \in \mathbb{R}} (b - q\left(\frac{Rb}{2} - 2\right)) (2 - Rb).$$

Solving the first order condition for a solution to this problem we get

$$b_2(R,q) = \frac{3Rq-2}{R(Rq-2)}, \;\; \theta_2(R,q) = \frac{3Rq-2}{2Rq-4} - 2.$$

Market clearing requires

$$\begin{aligned} b_1(R,q) + b_2(R,q) &= \frac{2}{R} + \frac{3Rq-2}{R(Rq-2)} = 0 \\ \theta_1(R,q) + \theta_2(R,q) &= \frac{4R+Rq+2}{2Rq} + \frac{3Rq-2}{2Rq-4} - 2 = 0. \end{aligned}$$

The solution to this system is $(R^*, q^*) = (1, 6/5)$. *The amounts that consumers borrow/lend are* $b_1(R^*, q^*) = -b_2(R^*, q^*) = 2$*, the number of units of the asset they buy/sell are* $\theta_1(R^*, q^*) = -\theta_2(R^*, q^*) = 3$*, and the equilibrium allocation is* $(x_1^*, y_1^*, z_1^*) = (33/10, 4, 0)$*, and* $(x_2^*, y_2^*, z_2^*) = (7/10, 0, 4)$.

Exercise 2. Two fishermen, Art (A) and Bob (B), have free access to a local lake. Each must choose how many days a week to fish, z_i. The total catch of fish obtained by each fisherman is $z_i(4-\bar{z})$, where z_i is the number of days $i \in \{A, B\}$ fishes, and $\bar{z} = (z_A + z_B)/2$ is the average number of days the two men fish a weak. Their preferences for fish and leisure are identical and are described by the utility function $u(x, y) = x + y$, where x is the fish consumed, and y is the number of days of leisure during the week. Naturally, each fisherman has 7 days a week for fishing and leisure activities.

(a) (10 points) Calculate how much time will each fisherman allocates to leisure and fishing.

(b) (10 points) Determine the socially optimal number of days the two men should fish, and the set of Pareto optimal allocations.

(c) (10 points) Assume that only Art has the right to fish in the lake, but he can sell the right to fish on any given day – with the implication that he cannot himself fish that day. The price p (in units of fish) of the right to fish one day is determined in a competitive market in which Art and Bob are the only participants. Determine the resulting allocation.

(a) In order to choose z_i each man solves the problem

$$\max_{z_i \in [0,7]} z_i \left(4 - \frac{z_i + z_j}{2}\right) + 7 - z_i$$

Solving the F.O.C. for an interior solution we get

$$z_i = R(z_j) = 3 - \frac{z_j}{2}.$$

Hence equilibrium is $z_1 = z_2 = z^*$, where

$$z^* = 3 - \frac{z^*}{2} = 2.$$

Thus, each men fishes 2 days a week for a total catch of fish equal to 4, and enjoys 5 days of leisure for a total utility equal to 9. This allocation of time is not Pareto optimal: for example, if both men were to reduce their fishing to just 1.5 days, then each men total catch of fish would be 3.75, and their days of leisure would be 5.5, for a total utility of 9.25, which makes both men better off.

(b) Let us identify the total number of days both men fish, $z \in [0, 14]$, that maximizes social welfare (i.e., the sum of both men utilities). This problem is

$$\max_{z \in [0,14]} z(4 - z/2) + (14 - z).$$

Solving the F.O.C. for an interior solution we get $z_S = 3$ days, for a total catch of fish of $3(4 - 3/2) = 7.5$ and a number of days left for leisure activities of 11. Hence, the set of Pareto optimal allocations is

$$P = \{[(x_1, y_1), (x_2, y_2)] = [(x, y), (7.5 - x, 11 - y), \ x \in [0, 7.5], \ y \in [0, 7]\}$$

(c) If Art supplies $s \in [0,7]$ permits, he should assume that Bob will fish during that number of days. Of course, Art may still fish for a number $z \in [0, 7-s]$ of days. Hence given the price p, Art decides his supply of permits and number of fishing days by solving the problem

$$\begin{aligned}&\max\nolimits_{s,z\in[0,7]} z\left(4-\frac{z+s}{2}\right)+ps+7-z\\&\text{s.t. } s+z\leq 7.\end{aligned}$$

The interior solution to this problem is

$$s(p)=6-4p,\ z_A(p)=2p.$$

Bob should anticipate this behavior and choose its demand of rights to solve

$$\max_{d\in[0,7]}\left(d\left(4-\frac{d+2p}{2}\right)-pd\right)+7-d.$$

The interior solution to this problem is

$$d(p)=3-2p.$$

For the market for rights to clear we must have $s(p)=d(p)$, i.e.,

$$6-4p=3-2p.$$

Hence $p^*=3/2$, and therefore $s(p^*)=d(p^*)=0$, $z_A(p^*)=3$. Thus, the equilibrium allocation is $[(x_1,y_1),(x_2,y_2)]=[(7.5,4),(0,7)]$, which is Pareto optimal. This is no surprise since the same allocation arise in the obsence of a market for permits.

Exercise 3. In a competitive insurance market there are two types of drivers, the *risky* ones (H), and the *prudent* ones (L), which are present in proportions $q \in (0,1)$ and $1-q$. The probability that a risky driver has an accident is $p_H = 1/2$, while this probability is only $p_L = 1/4$ for prudent drivers. All drivers have the same initial wealth, $W = 100$ monetary units, and preferences, represented by the Bernuilli utility function $u(x) = \ln x$. An accident generates a loss of 80 monetary units.

(a) (5 points) Calculate the policies that will be offered if a driver's type is observable.

(b) (10 points) Assume that a driver's type is private information (i.e., not observable), but by law insurance companies are restricted to offer a single full insurance policy. Determine the values of q for which companies will offer a policy that both types of drivers subscribe.

(c) (15 points) Calculate the separating policy menu, and verify whether, in the absence of legal constraints, it is a competitive equilibrium.

(a) In this case companies would offer the policy $(I_i, D_i) = (80p_i, 0)$ to drivers of type $i \in \{H, L\}$. That is, $(I_L, D_L) = (20, 0)$ and $(I_H, D_H) = (40, 0)$.

(b) The premium of the fair policy assuming that both types of drivers susbcribe is

$$\bar{I}(q) = qI_H + (1-q)I_2 = 40q + (1-q)20 = 20(1+q).$$

In order for drivers of type L to be willing to subscribe this policy we must have

$$\frac{1}{4}\ln(100-80) + \frac{3}{4}\ln 100 \leq \ln\left(100 - \bar{I}(q)\right),$$

that is,

$$\ln 20^{1/4}100^{3/4} \leq \ln\left(100 - \bar{I}(q)\right) \Leftrightarrow 20^{7/4} \leq 100 - \bar{I}(q)$$

or

$$q \leq \frac{100 - 20^{1/4}100^{3/4}}{20} - 1 \simeq 0.6563 := \bar{q}.$$

(c) The separating policies are $(I_H, D_H) = (40, 0)$, and $(\tilde{I}_L, \tilde{D}_L)$ satisfying

$$\tilde{I}_L = \frac{80 - \tilde{D}_L}{4}, \ \frac{1}{2}\ln\left(100 - \tilde{D}_L - I_L\right) + \frac{1}{2}\ln(100 - I_L) = \ln 60.$$

Writing $\tilde{D}_L = x$, we may write the second of these equations as (0)

$$\left(80 - \frac{3}{4}x\right)\left(80 + \frac{x}{4}\right) = 60^2.$$

Solving this equation we get

$$\tilde{D}_L = \frac{80}{3}\sqrt{37} - \frac{320}{3} \simeq 55.54, \tilde{I}_L = \left(80 - \left(\frac{80}{3}\sqrt{37} - \frac{320}{3}\right)\right)/4 \simeq 6.115.$$

For this menu to be a separating equilibrium the pooling policy cannot be preferred by the low risk drivers to the separating policy, that is

$$\ln\left(100-20(1+q)\right) \leq \frac{1}{4}\ln\left(100-\tilde{D}_L-\tilde{I}_L\right)+\frac{3}{4}\ln\left(100-\tilde{I}_L\right).$$

Substituting this inequality may be written as

$$100-20(1+q) \leq \left(100-\left(\frac{80}{3}\sqrt{37}-\frac{320}{3}\right)-\left(80-\left(\frac{80}{3}\sqrt{37}-\frac{320}{3}\right)\right)/4\right)^{\frac{1}{4}}\left(100-\left(\frac{80}{3}\sqrt{37}-\frac{320}{3}\right)/4\right)^{\frac{3}{4}}.$$

A numerical aproximation for the bound on q implies by this inequality is

$$q \geq 0.4828 = \hat{q}.$$

Exercise 4. Consider the contract design problem of a risk-neutral Principal who wants to hire an Agent. The Principal's revenue is a random variable $X(e)$ taking values $x_1 = 0$, $x_2 = 300$, and $x_3 = 600$ with probabilities that depends on the level of effort of the Agent, $e \in \{0, 1\}$. Specifically, these probabilities are $p_1(0) = 1/3$, $p_2(0) = 2/3$, and $p_1(1) = 1/3$, $p_2(1) = 1/3$, respectively. The Agent's preferences are represented by the Bernuilli utility function $u(w) = \sqrt{w}$, his reservation utility is $\underline{u} = 2$, and his costs of effort is $c(e) = 4e$.

(a) (10 points) Calculate the optimal contract assuming that effort is verifiable.

(b) (20 points) Calculate the optimal contract assuming that effort is *not* verifiable.

(a) Straightforward calculations yield $\mathbb{E}[X(0)] = 200$ and $\mathbb{E}[X(1)] = 300$. *Since effort is observable and the Principal is risk neutral, it is optimal to a offer a contract involving a fixed wage, i.e., a contract of the form* (e, w), *where* w *is a fixed wage. The wage* w *must satisfy*

$$\sqrt{w} = \underline{u} + c(e).$$

The contracts that satisfy this condition are $(e_0, w_0) = (0, 4)$ *and* $(e_1, w_1) = (1, 36)$. *The profit of the contract* (e_0, w_0) *is*

$$\mathbb{E}[X(0)] - 4 = 196,$$

while the profit of the contract (e_1, w_1) *is*

$$\mathbb{E}[X(1)] - 36 = 264.$$

Hence the optimal contract is (e_1, w_1).

(b) The contract $(e_0, w_0) = (0, 4)$ *remains optimal if the Principal wants the Agent to exert no effort, i.e.,* $e = 0$. *However, if the Principal wants the Agent to exert effort* $e = 1$, *then a fixed wage contract is not optimal, and the system of first order conditions identifying the optimal contract involves the participation and incentives constrains, as well as the equations*

$$\frac{1}{u'(w_i)} = \lambda + \mu\,(1 - l_i)\,,$$

where for $i \in \{1, 2, 3\}$,

$$l_i = p_i(0)/p_i(1).$$

Since $l_1 = (1/3)\,/\,(1/3) = 1$, $l_2 = (2/3)/(1/3) = 2$, *and* $l_3 = 0/(1/3) = 0$, *the system of first order conditions is*

$$\begin{aligned}
\frac{\sqrt{w_1}}{3} + \frac{\sqrt{w_2}}{3} + \frac{\sqrt{w_3}}{3} &= 2 + 4 \\
\frac{\sqrt{w_1}}{3} + \frac{\sqrt{w_2}}{3} + \frac{\sqrt{w_3}}{3} - 4 &= \frac{\sqrt{w_1}}{3} + \frac{2\sqrt{w_2}}{3} \\
2\sqrt{w_1} &= \lambda \\
2\sqrt{w_2} &= \lambda - \mu \\
2\sqrt{w_3} &= \lambda + \mu.
\end{aligned}$$

Using the change of variable $z_i = \sqrt{w_i}$, the system becomes linear and easy to solve. The unique solution is $\lambda = \mu = 12$, $z_1 = 6$, $z_2 = 0$, $z_3 = 12$. Hence the optimal wage contract if the Principal wants the Agent to exert effort $e = 1$ is $\hat{W} = (w_1.w_2, w_3) = (6^2, 0, 12^2)$. The profit generated by this contract is

$$\mathbb{E}[X(1)] - \mathbb{E}[\hat{W}(1)] = 300 - \frac{(6^2 + 12^2)}{3} = 240.$$

Since the profit generated by the contract $(e_0, w_0) = (0, 4)$ is 196, then the optimal contract is indeed $(1, \hat{W})$.

Exercise 5. Consider the contract design problem of a risk-neutral Principal who wants to hire an Agent. There are two types of agents, H and L, both with the same preferences, which are represented by the Bernuilli utility function $u(x) = x$, and the same reservation utilities $\underline{u} = 1$, but with different costs of effort, $v_i(e) = k_i e$ with $k_H = 1$ and $k_L = 2$. The fraction of agents of type H is $q \in (0, 1)$. The Principal's expected revenue is a function of the Agent's effort, $e \in [1, \infty)$, which is verifiable, and is given by $\bar{x}(e) = 8 \ln e$.

(a) (10 points) Calculate the contract the Principal will offer to each type of agent if types were observable. Illustrate your findings providing a graph of the Principal's demand of effort for each type of agent, as well as the supply of effort of each type of agent.

(b) (20 points) Identify the optimal menu of contracts the Principal will offer if the Agents' type is private information (that is, not observed by the Principal), assuming that the Principal wants to hire the agent whichever may be her type. Determine for which values of q the menu calculated is optimal.

Solution: With complete information the optimal contracts $[(e_H^*, w_H^*), (e_L^*, w_L^*)]$ *solve the systems*

$$\begin{aligned} \frac{8}{e_H} &= \frac{1}{1} \\ w_H &= e_H + 1 \ (PC_H) \end{aligned}$$

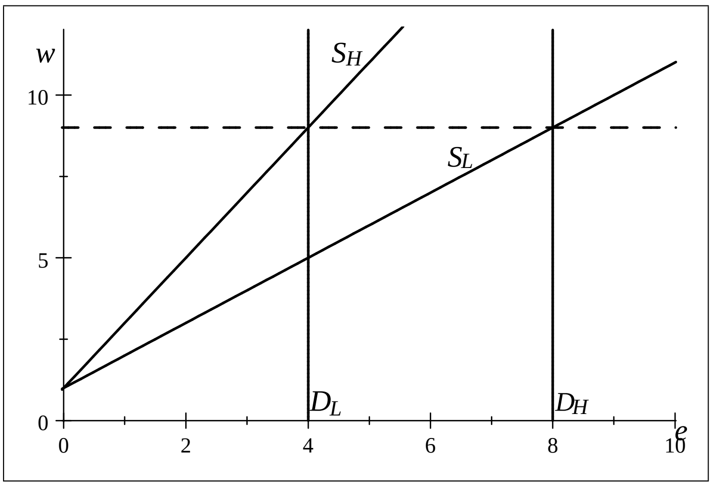
and

$$\begin{aligned} \frac{8}{e_L} &= \frac{2}{1} \\ w_L &= 2e_L + 1 \ (PC_L). \end{aligned}$$

Hence $[(e_H^*, w_H^*), (e_L^*, w_L^*)] = [(8, 9), (4, 9)]$. *Since*

$$\pi(e_H^*, w_H^*) = 8 \ln 8 - 9 > 0, \ \pi(e_L^*, w_L^*) = 8 \ln 4 - 9 > 0$$

This menu is optimal.



(b) (20 points) Identify the optimal menu of contracts the Principal will offer if the Agents' type is private information (that is, not observed by the Principal), assuming that the Principal wants to hire the agent whichever may be her type. Determine for which values of q the menu calculated is optimal.

Solution: As seen in class, the system of equations identifying the optimal menu with asymmetric information is:

$$\begin{aligned} \frac{8}{e_H} &= 1 \\ \frac{8}{e_L} &= 2+\frac{q}{1-q} \\ w_L &= 2e_L+1 \\ w_H-e_H &= w_L-e_L \end{aligned}$$

Solving the system we get $[(\tilde{e}_H,\tilde{w}_H),(\tilde{e}_L,\tilde{w}_L)] = \left[\left(8,\frac{26-17q}{2-q}\right),\left(\frac{8(1-q)}{2-q},\frac{18-17q}{2-q}\right)\right]$. *The Principal's profit with this menu is*

$$\begin{aligned} \tilde{\Pi} &= q\left(8\ln e_H - w_H\right) + (1-q)\left(8\ln e_L - w_L\right) \\ &= q\left(8\ln 8 - \frac{26-17q}{2-q}\right) + (1-q)\left(8\ln\left(\frac{8(1-q)}{2-q}\right) - \frac{18-17q}{2-q}\right) \\ &= 8(1-q)\ln\frac{8(1-q)}{2-q} + 8q\ln 8 - 9. \end{aligned}$$

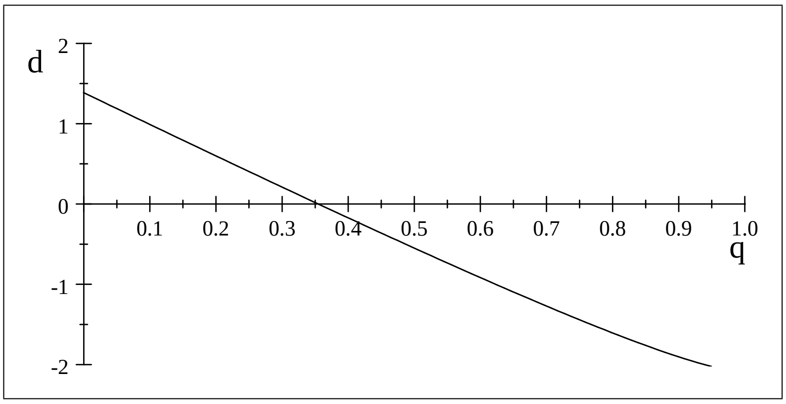
The profit of the Principal if he offers only the contract acceptable by agents of type H, $(e_H^*,w_H^*) = (8,9)$, is $q\pi\left(e_H^*,w_H^*\right) = q\left(8\ln 8 - 9\right)$. Hence

$$d(q) = \tilde{\Pi} - q\pi\left(e_H^*,w_H^*\right) = 8\left((1-q)\ln\frac{8(1-q)}{2-q} - q\ln 8\right).$$

Note that $d(0)=\ln 4>0$ and $d(1)=-\ln 8<0$, and

$$d'(q) = -\frac{1}{2-q}\left(3(2-q)\ln 2 + (2-q)\ln\frac{8(1-q)}{2-q} + 1\right)$$

is negative on $(0,1)$. Hence there is a threshold value $\bar{q}\in(0,2)$ such that $d(q)\geq 0 \Leftrightarrow q\leq\bar{q}$. This value is $\bar{q}\simeq 0.354\,78$ – see the graph below.



Exercise 6. In a competitive labor market there are two types of workers, H and L. The expected revenue of a firm that hires a worker of type $i \in \{H, L\}$ with a level of education $y \in \mathbb{R}_+$ is $\bar{x}_i(y) = a_i + \sqrt{y}$, where $a_H = 2$ and $a_L = 1$. Workers choose their levels of education before entering the labor market. Firms observe the workers' level of education, but not their types, and make a wage offer. All workers have the same preferences, represented by the utility function $u(w) = w$, and reservation utility is $\underline{u} = 0$, but different costs of education, given by $c_H(y) = y/2$ and $c_L(y) = y$, respectively. Firms maximize expected profits. (Of course, in a competitive equilibrium firms' profits are zero.)

(a) (15 points) Compute the most efficient separating PBNE (that is, the PBNE in which a worker's education signals her type). (Show your solution graphically first, and then do the algebra.)

(b) (15 points) Assume that the fraction of workers of type L is $q \in (0, 1)$. Compute the pooling (no signaling) PBNE in which the workers level of education $\hat{y}$ maximizes the surplus,

$$S(y) = q(\bar{x}_L(y) - c_L(y)) + (1 - q)\,(\bar{x}_H(y) - c_H(y)),$$

and identify the values of q for which this PBNE Pareto dominates the PBNE identified in (a).

In a competitive equilibrium $w_i = \bar{x}_i(y_i)$ for $i \in \{H, L\}$. Note that $y_L^* = 1/4$ maximizes $\bar{x}_L(y)$, and $y_H^* = 1$ maximizes $\bar{x}_H(y)$. In the most efficient PBNE we have $y_L = y_L^*$, and $y_H := \bar{y}$ is the closest value to $y_H^* = 1$ in $[1, \infty)$ such that an L worker is indifferent between signaling y_L^* or $\bar{y}$; that is,

$$w_L^* - c_L(y_L^*) \geq w_H^* - c_L(\bar{y}).$$

Substituting we obtain

$$(1 + \frac{1}{2}) - \frac{1}{4} \geq (2 + \sqrt{\bar{y}}) - \bar{y},$$

i.e.,

$$-\frac{3}{4} \geq \sqrt{\bar{y}} - \bar{y}.$$

Hence

$$\bar{y} = \frac{9}{4},$$

and

$$w_H = 2 + \sqrt{\frac{9}{4}} = \frac{7}{2}.$$

The equilibrium payoffs are

$$u_L^* = w_L^* - c_L(y_L^*) = \frac{3}{2} - \frac{1}{4} = \frac{5}{4}$$

and

$$u_H^* = w_H^* - c_H(\bar{y}) = \frac{7}{2} - \frac{9}{8} = \frac{19}{8}.$$

(b) We have

$$\begin{aligned} W(y) &= q(1+\sqrt{y}-y)+(1-q)\left(2+\sqrt{y}-\frac{y}{2}\right) \\ &= 2-q+\sqrt{y}-\frac{1+q}{2}y, \end{aligned}$$

Therefore, the solution to the equation

$$W'(y)=\frac{1}{2\sqrt{y}}-\frac{1+q}{2}=0$$

is

$$\hat{y}(q)=\frac{1}{(1+q)^2}.$$

The pooling wage is

$$\begin{aligned} \hat{w}(q) &= q\left(1+\frac{1}{1+q}\right)+(1-q)\left(2+\frac{1}{1+q}\right) \\ &= 2+\frac{1}{1+q}-q. \end{aligned}$$

Thus,

$$\hat{u}_L(q)=2+\frac{1}{1+q}-q-\frac{1}{(1+q)^2}.$$

Obviously, a type L worker prefers this outcome. (It is easy to show that $d\hat{u}_L(q)/dq<0$ in $(0,1)$ and $\hat{u}_L(1)=5/4$.)

For a type H weaker we have

$$\hat{u}_H(q)=2+\frac{1}{1+q}-q-\frac{1}{2\,(1+q)^2}=\frac{19}{8}.$$

Hence a type H worker prefers the pooling equilibrium whenever

$$\hat{u}_H(q)>\frac{19}{8},$$

which requires

$$q\geq 0.12.$$