Exercise 1. A good of two qualities, high (H) and low (L), is traded in competitive markets in which each seller has a single unit and each buyer wants to buy a single unit. There are n_H sellers with a unit of high quality whose opportunity cost is c_H euros, n_L sellers with a unit of low quality whose opportunity cost is c_L euros, and n buyers who value a unit of high quality in u_H euros and a unit of low quality in u_L euros. Assume that $u_H > c_H > u_L > c_L$.

(a) Suppose that quality is observable. Calculate the competitive prices for the cases $n > n^H + n^L$ and $n < n_H + n_L$. Discuss if these competitive equilibria generate the maximum surplus. (If you find it helpful, assume $u_H = 10$, $c_H = 7$, $u_L = 5$, $c_L = 0$, $n_H = 1$, $n_L = 1$ and $n \in \{1, 3\}$.)

(b) Now suppose that quality is not observable and both qualities trade in the same market. Also assume that $n = n_H + n_L$. Represent the supply and demand schedules in the plane (q, p) and calculate the competitive equilibria of this market when the expected value of a random unit,

$$u(n^H, n^L) = \frac{n_H}{n_H + n_L} u_H + \frac{n_L}{n_H + n_L} u_L,$$

is greater than c^H, and when it is less than c^H. (If you find it helpful, use the parameter values suggested in part (a), and consider the cases $n_L = 1$ and $n_L = 2$.)

PARAMETERS: $\quad u_H > c_H > u_L > c_L$

$$n_B > \max\{n_H, n_L\}$$

CE WHEN QUALITY IS OBSERVABLE:

DIFFERENCIATED MARKETS FOR <u>HIGH</u> AND <u>LOW</u> QUALITY.

LET τ: $\quad u_\tau - c_\tau = \max\{u_H - c_H, u_L - c_L\}$.

<u>CE</u>:

1 $\quad n_B < n_H + n_L$

- n_B UNITS TRADE
- ALL τ-QUALITY UNITS TRADE
- $n_B - n_H - n_L$ UNITS OF THE QUALITY $\tau' \neq \tau$ <u>DO NOT</u> TRADE.
- $P_{\tau'} = c_{\tau'}$
- $u_\tau - P_\tau = u_{\tau'} - c_{\tau'}$

THE MAXIMUM SURPLUS IS REALIZED.

(2) $n_B > n_H + n_L$

- Buyers' surplus is zero $\Rightarrow$ $P_H = u_H$, $P_L = c_L$
- All units trade $\Rightarrow$ Maximum surplus is realized.

(3) $n_B = n_H + n_L$

- Multiple CE exist. Specifically, the set of CE prices is

$$\{ (P_H, P_L) \in [c_H, u_H] \times [c_L, u_L] \mid u_H - P_H = u_L - P_L \}$$

- All units trade $\Rightarrow$ Maximum surplus is realized.

(a) $n_H = n_L = 1$, $n \in \{1, 2, 3\}$.

$n_B = 1 \implies (p_H, p_L) \in [0, 7] \times [0, 2]$; e.g., $(p_H^*, p_L^*) = (5, 1)$

$n_B = 2 \implies \{ (p_H, p_L) \in [7, 10] \times [0, 5] \mid 10 - p_H = 5 - p_L \}$;

e.g., $(p_H^*, p_L^*) = (8, 3)$

$n_B = 3 \implies p_H^* = u_H = 10$, $p_L^* = u_L = 5$.

UNOBSERVABLE QUALITY: ALL UNITS TRADE IN THE SAME MARKET. THE QUALITY IS OBSERVED ONL UPON PURCHASE.

LET US WRITE

$$\bar{u} := \frac{n_H}{n_H + n_L} u_H + \frac{n_L}{n_H + n_L} u_L$$

FOR THE BUYERS' EXPECTED VALUE OF A RANDOM UNIT WHEN ALL SELLERS SUPPLY. (WE ASSUME THAT BUYERS ARE RISK-NEUTRAL.)

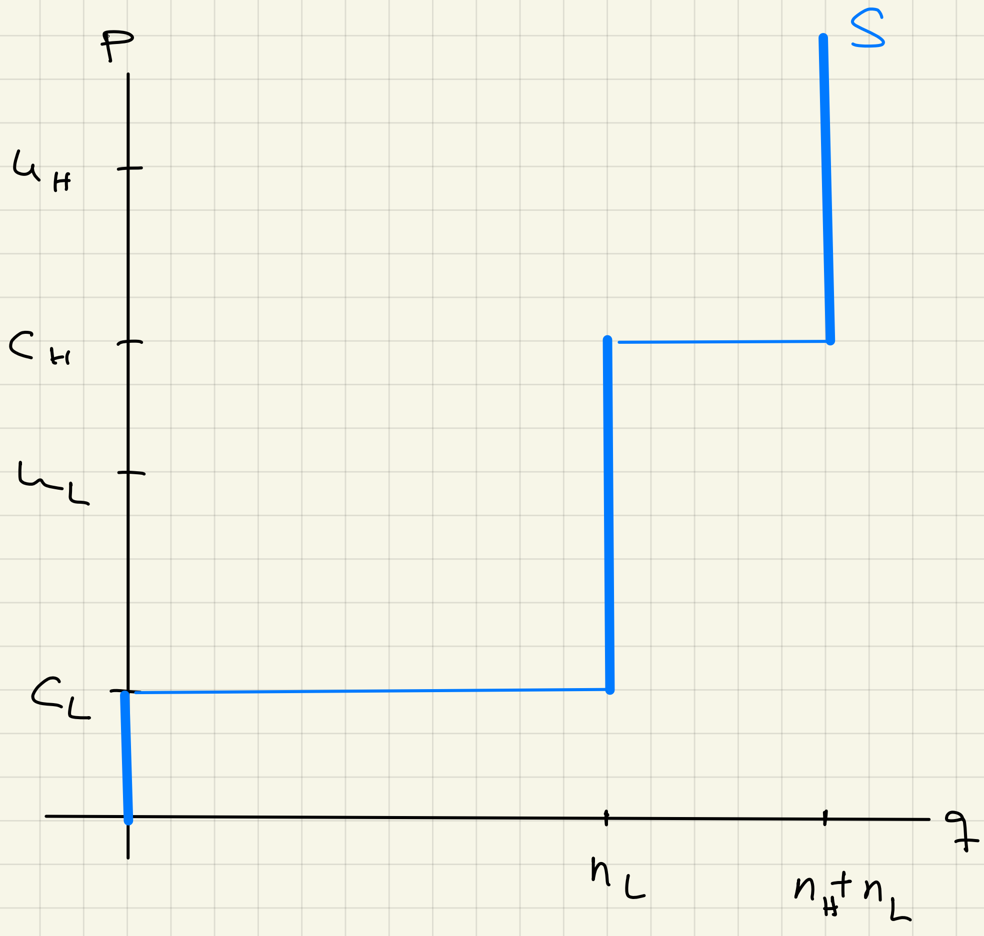
WHAT ARE THE SUPPLY AND DEMAND

OF THIS GOOD OF RANDOM QUALITY?

Supply: Easy!

Demand?

Buyers, who know the costs of sellers, can make inferences about the quality that are supplied at each price.



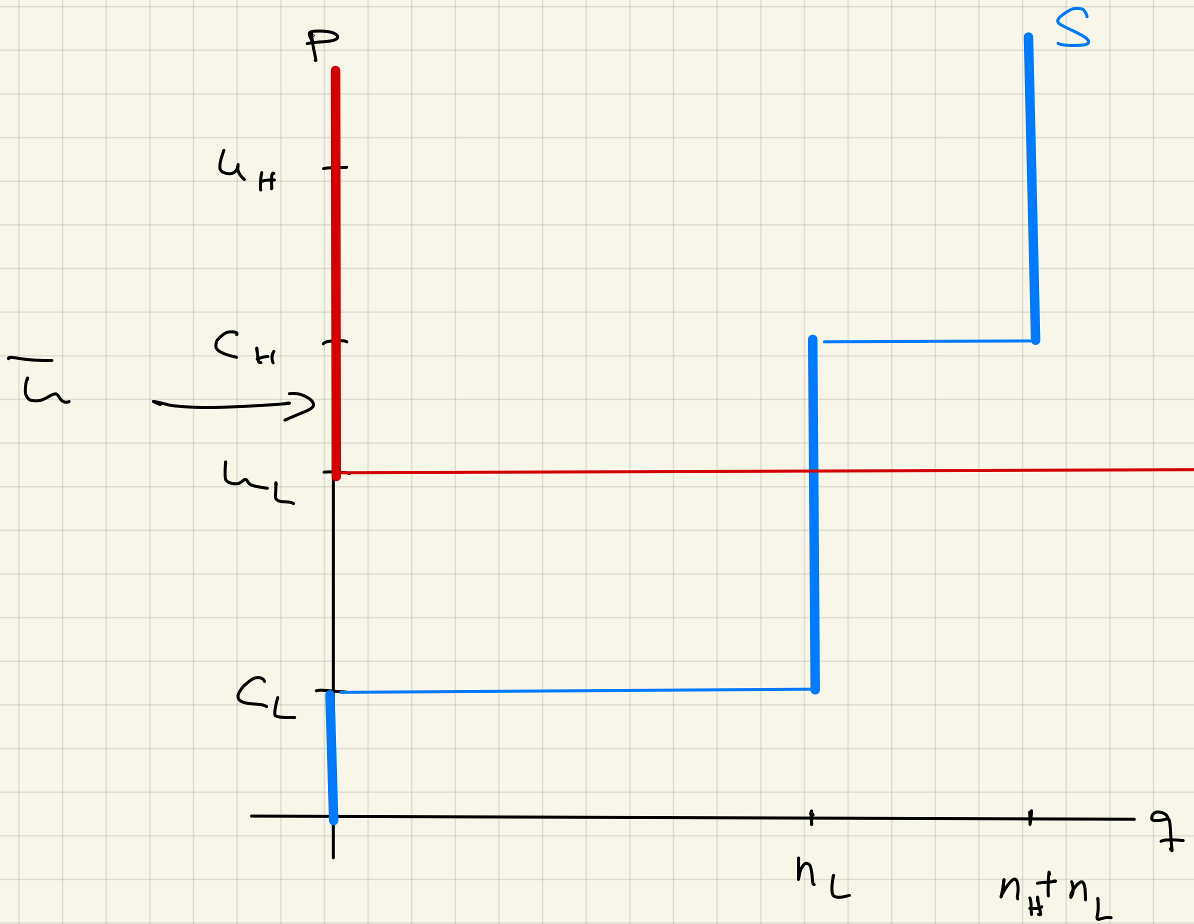
$$E[u \mid p] = \begin{cases} \bar{u} & \text{if } p > c_H \\ u_L & \text{if } p \in (c_L, c_H) \\ ? & p < c_L \end{cases}$$

$$D(p) = \begin{cases} n_B & \text{if } E[u \mid p] > p \\ \{0, 1, \ldots, n_B\} & \text{if } E[u \mid p] = p \\ 0 & \text{if } E[u \mid p] < p \end{cases}$$

(i) $\bar{u} < c_H$

Since $n_B > n_L$,

$$\boxed{p^* = u_L}$$



THE CE IS INEFFICIENT!

(b) (c)

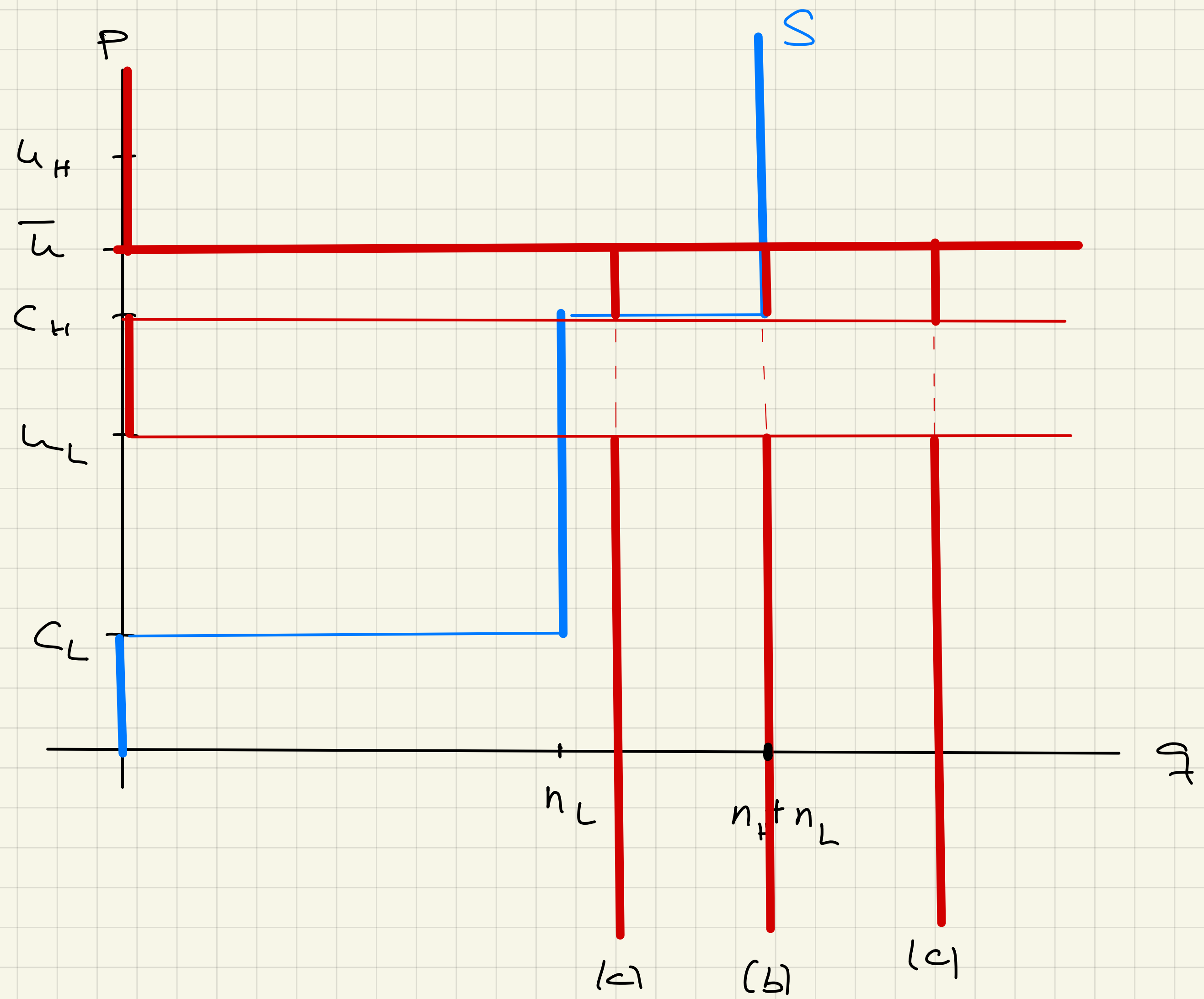
(ALL HIGH-QUALITY SELLERS AND SOME BUYERS DO NOT TRADE EVEN THOUGH THERE ARE GAINS TO TRADE — RECALL THAT $n_B > n_L$.)

(ii) $\bar{u} > c_H$

(a) $n_B < n_H + n_L$

(b) $n_B = n_H + n_L$

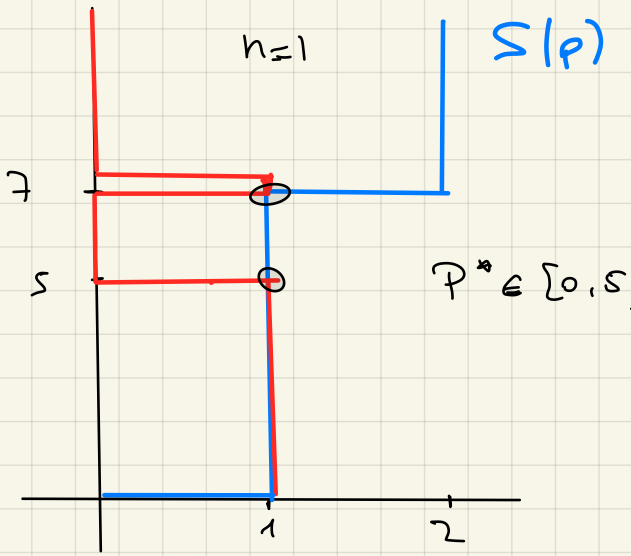
(c) $n_B > n_H + n_L$



THERE EFFICIENT AS WELL AS INEFFICIENT CE!

(b) $n^H = n^L = 1$:

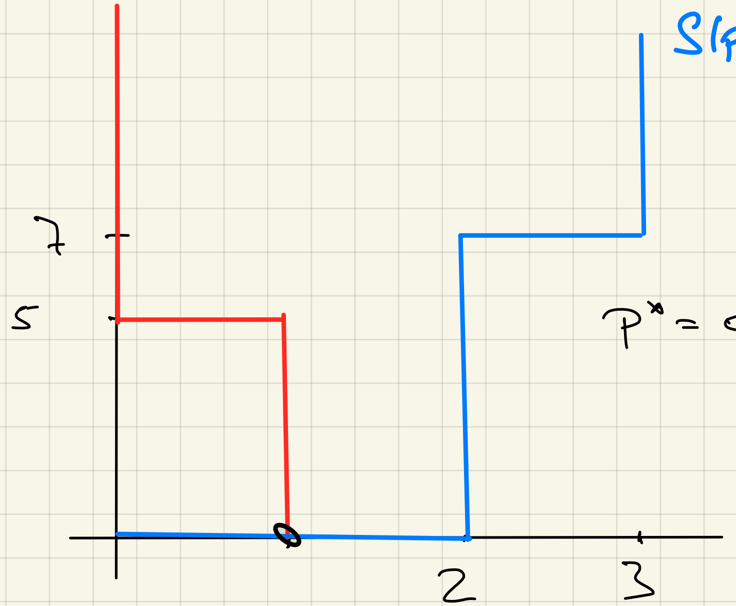
$$E[u] = \frac{u^H + u^L}{2} = 7.5$$



$P^* \in [0, 5]$

$n^H = 1,\ n_L = 2$:

$$E[u] = \frac{u^H + 2u^L}{3} = \frac{20}{3} < 7$$



$P^* = 0$

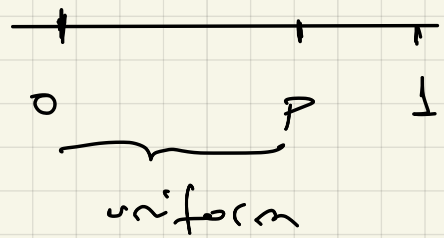
A more extreme example.

A measure 1 of sellers. The quality of the good supplied by the seller $s \in [0,1]$ is $q(s) = s$, and its opportunity cost is $c(s) = s$.
Buyers' value of quality $s \in [0,1]$ is $u(s) = \alpha s$, $\alpha \in (1,2)$.

CE

Supply: $p \in \mathbb{R}_+$, $S(p) = \min\{p, 1\}$.

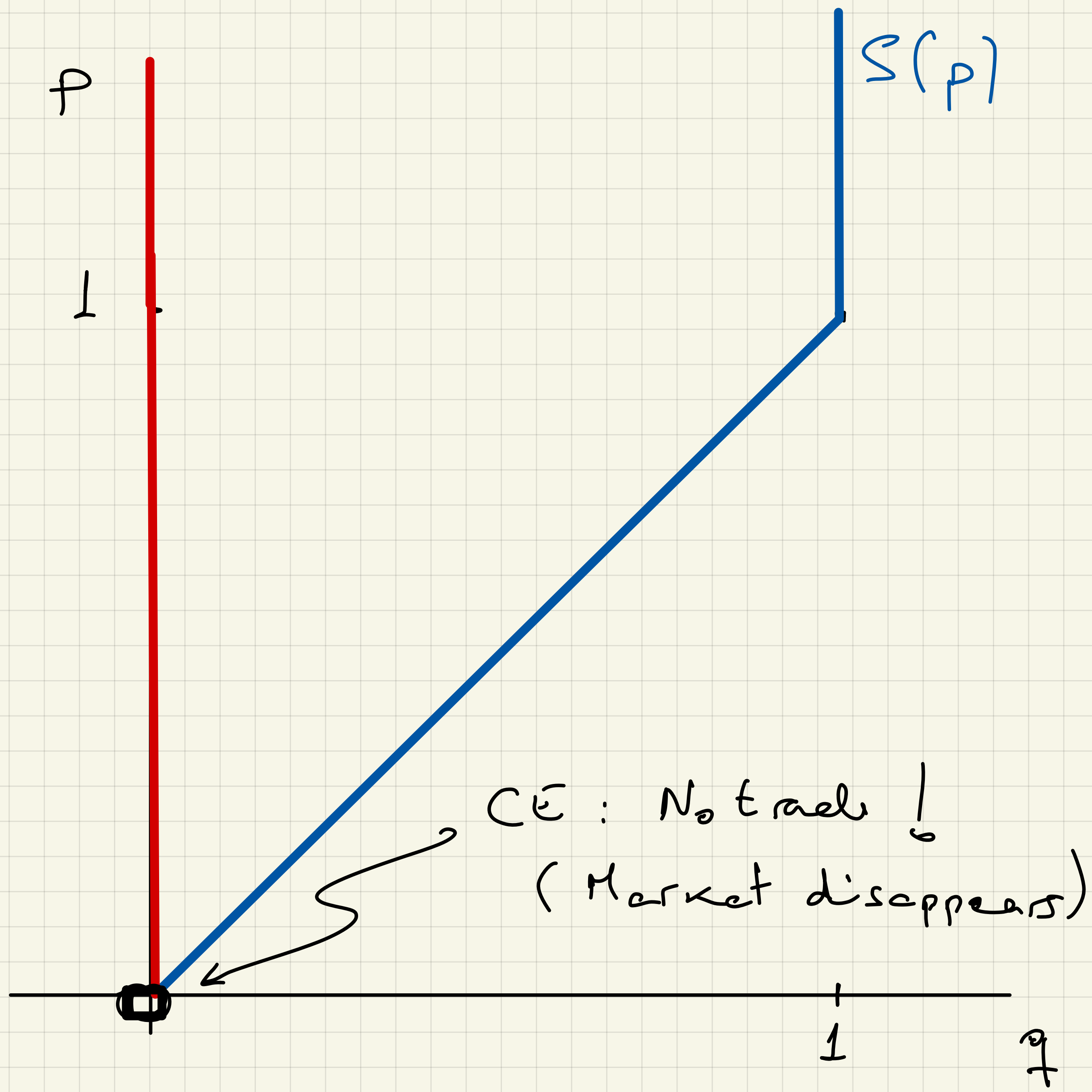
Demand: $D(p) = \begin{cases} 0 & \text{if } p > E[u(S) \mid S \le p] \\ 1 & \text{otherwise.} \end{cases}$



$\mathbb{E}[u(S) \mid S \le p]$?

If $p > 1$, $E[u(S) \mid S \le p] = \alpha E[S \mid S \le p] = \alpha/2 < 1 < p \Rightarrow D(p) = 0$

If $p \le 1$: $E[u(S) \mid S \le p] = \alpha E[S \mid S \le p] = \alpha\left(\frac{p}{2}\right) < p \Rightarrow D(p) = 0$


p
1
S(p)
CE : No trade !
(Market disappears)
1
q

Exercise 2. Consider a market for used cars whose qualities, indexed by the sellers' cost, are uniformly distributed in the interval $[2, 6]$. Buyers are risk-neutral and value each quality 20% more than sellers. Naturally, each seller knows the quality of the good he sells, but quality is not observable to buyers prior to purchase. Assume that there are more buyers than sellers.

(a) Determine the market supply and the average quality of the cars offered at each price.

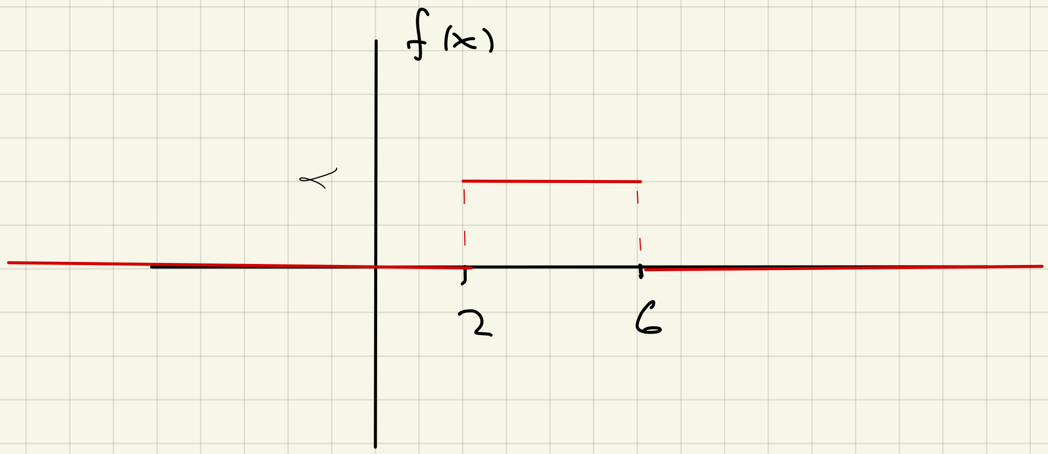
(b) Calculate the market equilibrium.

SUPPLY

$$s(p) = \begin{cases} 0 & p < 2 \\ p-2 & p \in [2,6] \\ 4 & p > 6. \end{cases}$$

DEMAND : MORE INVOLVED ----

$$X \sim U[2,6]$$



THUS, THE PDF OF X IS, FOR $x \in \mathbb{R}$,

$$f_X(x) = \begin{cases} \alpha & x \in [2,6] \\ 0 & x \notin [0,6], \end{cases}$$

WHERE α SATISFIES

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_2^6 \alpha\,dx = \alpha\,[x]_2^6 = 4\alpha \iff \alpha = \frac{1}{4}.$$

AND THE CDF OF X IS, FOR $x \in \mathbb{R}$,

$$F_X(x) = \int_{-\infty}^{x} f_X(z)\,dz = \begin{cases} 0 & \text{IF } x < 2 \\ \displaystyle\int_2^x \frac{dz}{4} = \frac{[z]_2^x}{4} = \frac{x-2}{4} & \text{IF } x \in [2,6] \\ 1 & \text{IF } x > 6. \end{cases}$$

LET $p \in [2, 6]$. WHAT IS THE DISTRIBUTION OF $X / X \le p$?

BAYES' RULE: LET (Ω, P_r) BE A PROBABILITY MODEL, AND LET A AND B ANY TWO EVENTS, $A, B \subset \Omega$, WHERE $P_r(B) > 0$. THEN

$$P_r(A / B) = \frac{P_r(A \cap B)}{P_r(B)}.$$

LET US APPLY BAYES' RULE TO CALCULATE $f_{X/X \le p}$.

LET $p \in (2, 6)$, AND $x \in \mathbb{R}$:

$$F_{X / X \le p}(x) = P_r(X \le x / X \le p) = \frac{P_r(X \le x, X \le p)}{P_r(X \le p)}$$

SINCE

$$P_r(X \le x, X \le p) = \begin{cases} F_X(x) & \text{IF } x < p \\ \\ F_X(p) & \text{IF } x \ge p, \end{cases}$$

Then

$$F_{X|X\le p}(x) = \begin{cases} \dfrac{F_X(x)}{F_X(p)} & \text{if } x \le p \\[2ex] \dfrac{F_X(p)}{F_X(p)} & \text{if } x \le p \end{cases} = \begin{cases} 0 & \text{if } x < 2 \\[1ex] \dfrac{x-2}{p-2} & \text{if } x \in [2, p] \\[1ex] 1 & \text{if } x > p. \end{cases}$$

Hence

$$f_{X|X\le p}(x) = F'_{X|X\le p}(x) = \begin{cases} 0 & \text{if } x < 2 \\[1ex] \dfrac{1}{p-2} & \text{if } x \in [0, p] \\[1ex] 0 & \text{if } x > p, \end{cases}$$

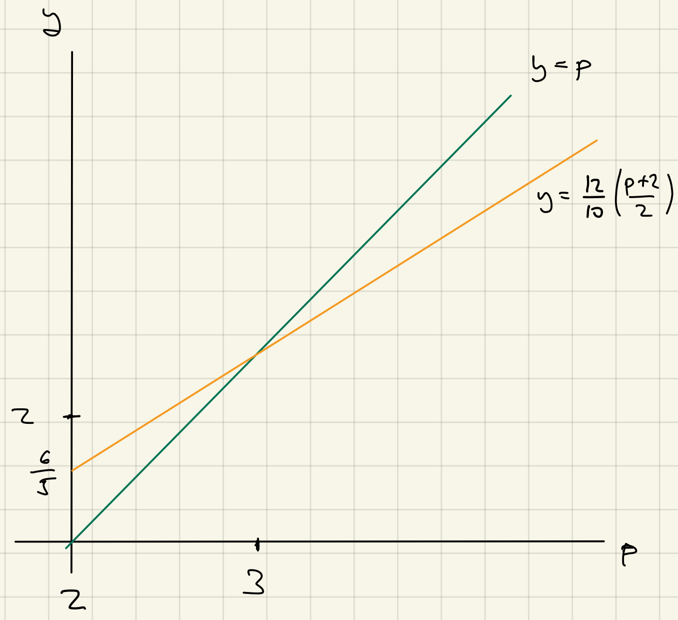
and

$$E[X|X\le p] = \int_{-\infty}^{\infty} x f_{X|X\le p}(x)\,dx = \int_2^p \frac{x}{p-2}\,dx = \frac{\left[\frac{x^2}{2}\right]_2^p}{p-2}$$

$$= \frac{1}{2}\,\frac{p^2-4}{p-2} = \frac{1}{2}\,\frac{\cancel{(p-2)}(p+2)}{\cancel{(p-2)}} = \frac{p+2}{2}.$$

A BUYER WILL BY AT A PRICE $p \in [2, 6]$ iff

$$\frac{12}{10} E\left[X \mid X \le p\right] = \frac{12}{10}\left(\frac{p+2}{2}\right) \ge p \iff p \le 3$$



HENCE DEMAND FOR $p \in \mathbb{R}_+$ IS

$$D(p) = \begin{cases} n_B & \text{IF } p < 3 \\ [0, n_B] & \text{IF } p = 3 \\ 0 & \text{IF } p > 3. \end{cases}$$

AND THE CE PRICE IS $p^* = 3$, AND ONLY THE QUALITIES IN THE INTERVAL $[2, 3]$ TRADE.