Final Exam
(May 27, 2016)

Answer the following three exercises:

Exercise 1. In an economy that extends over two periods, today and tomorrow, there are two consumers, A and B, and a single perishable good. The state of nature tomorrow can either be H or L. Consumers' preferences over consumption today and tomorrow are represented by the utility functions $u_A(c_0, c_H, c_L) = c_0 c_H$, and $u_B(c_0, c_H, c_L) = c_0(c_H + c_L)$, and their endowments are $(\omega_0^A, \omega_H^A, \omega_L^A) = (8, 0, 0)$ and $(\omega_0^B, \omega_H^B, \omega_L^B) = (0, 4, 4)$.

(a) (20 points) Assume that today there are markets for consumption today, consumption tomorrow at state H, and consumption tomorrow at state L, in which the prices are $p_0 = 1$, $p_H > 0$ and $p_L > 0$, respectively. Write down the problem each consumer faces, calculate the system of first order conditions identifying an interior solution, and obtain the consumers' demands. Then calculate the (Arrow-Debreu) competitive equilibrium (CE) price vector and allocation. (Hint. Consumer A demands zero units of c_L. As for consumer B, calculate the marginal rate of substitution of any two goods (MRS), and note that $MRS_{c_H c_L} = 1$, that is, c_H and c_L are perfect substitutes; therefore $p_H > p_L \Rightarrow c_H = 0$, and $p_H < p_L \Rightarrow c_L = 0$. In order to calculate the CE prices, show that price vectors such that either $p_H > p_L$ or $p_H < p_L$ cannot be CE. Then proceed to calculate the CE price vector.)

Solution: Each consumer $i \in \{A, B\}$ solves the problem

$$\begin{array}{rl} & \max\limits_{[(c_0, c_H, c_L)] \in \mathbb{R}^3_+} u_i(c_0, c_H, c_L) \\ s.t. & c_0 + p_H c_H + p_L c_L \leq \omega_0 + p_H \omega_H + \omega_L p_L. \end{array}$$

The first order conditions for an interior solution are

$$\begin{aligned} MRS^i_{c_0 c_H} &= \frac{1}{p_H} \\ MRS^i_{c_0 c_L} &= \frac{1}{p_L} \\ c_0 + p_H c_H + p_L c_L &= \omega_0 + p_H \omega_H + \omega_L p_L. \end{aligned}$$

For consumer A, if $p_L > 0$, then

$$c_L^A(p_H, p_L) = 0.$$

Hence, since $MRS^A_{c_0 c_H} = c_H / c_0$, and $\omega_0^A = 8$, $\omega_H^A = \omega_L^A = 0$, his demands of c_0 and c_H are identified by the system of equations

$$\begin{aligned} \frac{c_H}{c_0} &= \frac{1}{p_H} \\ c_0 + p_H c_H &= 8. \end{aligned}$$

Solving the system we get

$$
\begin{aligned}
c_0^A(p_H, p_L) &= 4 \\
c_H^A(p_H, p_L) &= \frac{4}{p_H}.
\end{aligned}
$$

For consumer B, $MRS^B_{c_0 c_H} = MRS^B_{c_0 cL} = (c_H + c_L)/c_0$, *and* $MRS^B_{c_H c_L} = 1$, *and* $\omega_0^B = 0$, $\omega_L^B = \omega_H^B = 4$. *Therefore the system of first order conditions is*

$$
\begin{aligned}
\frac{c_H + c_L}{c_0} &= \frac{1}{p_H} \\
\frac{c_H + c_L}{c_0} &= \frac{1}{p_L} \\
c_0 + p_H c_H + p_L c_L &= 4(p_H + p_L).
\end{aligned}
$$

The first two equations imply $p_H = p_L$, *which is consistent with the fact that* c_H *and* c_L *are perfect substitutes. For prices not satisfying this equation, the solution of the consumers problem is not interior: If* $p_H > p_L$, *then* $c_H^B(1, p_H, p_L) = 0$, *and the system above yields*

$$
\begin{aligned}
c_0^B(1, p_H, p_L) &= 2(p_H + p_L) \\
c_L(1, p_H, p_L) &= \frac{2(p_H + p_L)}{p_L}.
\end{aligned}
$$

Likewise, if $p_H < p_L$, *then* $c_L^B(p_H, p_L) = 0$, *and the system above yields*

$$
\begin{aligned}
c_0^B(1, p_H, p_L) &= 2(p_H + p_L) \\
c_H(1, p_H, p_L) &= \frac{2(p_H + p_L)}{p_H}.
\end{aligned}
$$

If $p_H = p_L$, *then the system yields*

$$
\begin{aligned}
c_0^B(1, p_H, p_L) &= 2(p_H + p_L) \\
c_H(1, p_H, p_L) + c_L(1, p_H, p_L) &= \frac{2(p_H + p_L)}{p_H}.
\end{aligned}
$$

Let us show that in a CE $p_H \leq p_L$. *Suppose* $p_H > p_L$; *then market clearing requires*

$$
\begin{aligned}
c_H^A(1, p_H, p_L) + c_H^B(1, p_H, p_L) &= \frac{4}{p_H} + 0 = 4 \\
c_0^A(1, p_H, p_L) + c_0^B(1, p_H, p_L) &= 4 + \frac{2(p_H + p_L)}{p_H} = 8,
\end{aligned}
$$

i.e., $p_H = p_L = 1$, *which is a contradiction.*

Let us show that in a CE $p_H \geq p_L$. Suppose $p_H < p_L$; then market clearing requires

$$c_L^A(1, p_H, p_L) + c_L^B(1, p_H, p_L) = 0 + 0 = 4,$$

which is a contradiction.

Hence $p_H^* = p_L^*$. Since the market clearing conditions

$$c_0^A(1, p_H^*, p_L^*) + c_L^B(1, p_H^*, p_L^*) = 4 + 2\,(p_H^* + p_L^*) = 8,$$

must hold, then $p_H^* + p_L^* = 2$. Hence $p_H^* = p_L^* = 1$. It is easy to check that $(p_0^*, p_H^*, p_L^*) = (1, 1, 1)$ is indeed a CE: Substituting in the demand of consumer A calculated above we get

$$(c_H^A(1, 1, 1), c_H^A(1, 1, 1), c_L^A(1, 1, 1)) = (4, 4, 0).$$

And because c_H^B and c_L^A, are perfect substitutes, $c_H^B(1, 1, 1) + c_L^B(1, 1, 1) = 4$. Thus,

$$(4, 0, 4) \in (c_0^B(1, 11,), c_H^B(1, 1, 1), c_L^B(1, 1, 1)),$$

that is, $(4, 0, 4)$ solves consumer B's problem when prices are $(1, 1, 1)$. Thefore

$$(p_0^*, p_H^*, p_L^*) = (1, 1, 1)$$

is a CE price vector leading to the CE allocation

$$[(c_0^A, c_H^A, c_L^A), (c_0^B, c_H^B, c_L^B)] = [(4, 4, 0), (4, 0, 4)].$$

Obviously this allocation is Pareto optimal.

(b) (10 points) Now suppose that the only market available is a credit market. Normalize the price of the consumption good at each date and state to be 1, and denote by r the interest rate. Then write down the problem each consumer solves (including his budget constrains). (Denote by $b_i(r)$ how much consumer i borrows at interest rate r.) Calculate the CE interest rate, r^*, how much each consumer borrows, and how much he consumes at each date and state. Determine whether the competitive allocation is Pareto optimal.

Solution: Each consumer $i \in \{A, B\}$ solves the problem

$$\begin{array}{rrcl} & \multicolumn{3}{c}{\max_{[(c_0,c_H,c_L),b]\in\mathbb{R}^3_+\times\mathbb{R}} u_i(c_0, c_H, c_L)} \\ s.t. & c_0 & \leq & \omega_0 + b \\ & c_H & \leq & \omega_H - (1+r)b \\ & c_L & \leq & \omega_L - (1+r)b. \end{array}$$

For consumer A, this problem is equivalent to

$$\max_{b\in\mathbb{R}} (8+b)(-(1+r)b).$$

A solution to this problem solve

$$-(1+r)(8+2b) = 0.$$

Hence

$$b_A(r) = -4.$$

For consumer B, his problem is equivalent to

$$\max_{b\in\mathbb{R}} b[(4-(1+r)b) + 4 - (1+r)b].$$

A solution to this problem solve

$$8 - 4(1+r)b = 0.$$

Hence

$$b_B(r) = \frac{2}{1+r}.$$

Hence market clearing implies

$$b_A(r) + b_B(r) = -4 + \frac{2}{1+r} = 0,$$

which implies

$$r^* = -\frac{1}{2}.$$

The resulting allocation is

$$[(c_0^A, c_H^A, c_L^A), (c_0^B, c_H^B, c_L^B)] = [(4,2,2),(4,2,2)].$$

This allocation is not Pareto optimal because the allocation $[(4,3,0),(4,1,4)]$ is Pareto superior. (The CE allocation identified in (a) is only weakly Pareto superior as only consumer A is better off than in the allocation here.)

(c) (10 points) Now suppose that in addition to a credit market there is a market for a security y that opens also today. The security y pays one unit of consumption tomorrow if the state is H, and nothing otherwise. Denote by q the market price of this security (in units of consumption today). Determine the competitive equilibrium interest rate and security price, (r^*, q^*). (Hint. One may argue that this economy is equivalent to the Arrow-Debreu economy described is part (a), and therefore a CE of this economy corresponds to a CE of the economy in (a). If you write the single consolidated budget constrain (involving the three consumption goods c_0, c_H, and c_L) of a consumer in this present economy, you will be able to identify the one to one mapping between the Arrow-Debreu CE prices calculated in (a), $(1, p_H^*, p_L^*)$, and the interest rate and security price in the CE of this economy, (r^*, q^*).

Solution: In this economy a consumer's budget constrains, which are satisfied with equality, are

$$\begin{aligned} c_0 &= \omega_0 + b - qy \\ c_H &= \omega_H - (1+r)b + y \\ c_L &= \omega_L - (1+r)b. \end{aligned}$$

Substituting into the first equation

$$b = (\omega_L - c_L) \,/\, (1+r)$$

from the third equation, and

$$y = (1+r)\, b - (\omega_H - c_H) = (\omega_L - c_L) - (\omega_H - c_H)$$

from the second equation, and rearranging we get

$$c_0 + \left(\frac{1}{1+r} - q\right) c_H + q c_L = \omega_0 + \left(\frac{1}{1+r} - q\right) \omega_H + q \omega_L.$$

This constrain is the same as that of part (a), with

$$p_H = \left(\frac{1}{1+r} - q\right), \; p_L = q.$$

Since this economy is equivalent to that described is part (a), the CE allocation is the same. Hence in the competitive equilibrium of the economy

$$q^* = p_L^* = 1,$$

and

$$\left(\frac{1}{1+r} - q\right) = p_H^* = 1,$$

that is

$$r^* = \frac{1}{2}.$$

Exercise 2. An economy consist of 3 individuals who only care about their consumption. There is a technology freely available that allows to produce K units of consumption good for each hour of labor used as input. The parameter K represents the *state of knowledge*; specifically, $K = \sum_{i=1}^{n} z_i$, where z_i is the time individual i spends improving the technology. Each individual is endowed with 12 hours that he can use to produce consumption good and/or improve the technology.

(a) (5 points) Identify the Pareto optimal state of knowledge K^*, and the corresponding per capita consumption in this economy.

Solution. Given the symmetry of the problem, let us focus on a symmetric solution to the social planner's problem of maximizing per capita consumption. If each agent contributes z to improving the technology, then $K(z) = 3z$, the per capita consumption would be $(3z)\,3\,(12 - z)\,/3 = 3z\,(12 - z)$. Then the optimal contribution z^* solves

$$\max_{z\in[0,12]} \; 3z\,(12 - z)\,.$$

Hence the optimal per capita allocation of time to be spent improving the technology is $z^* = 6$, the optimal state of knowledge is $K(z^*) = 18$, and the per capita consumption is

$$c^* = 18(12 - 6) = 108.$$

(b) (10 points) Determine the state of knowledge that would result if each individual decides how much time to spend improving the technology attending to his own interests. (You may assume that the equilibrium is symmetric; that is, you need not prove it.)

Solution. Each individual i chooses how much time he spends improving the technology z_i by solving the problem

$$\max_{z\in[0,12]} \; (z_i + z_{-i})\,(12 - z_i)\,,$$

where $z_{-i} = \sum_{j\neq i} z_j$. Hence

$$z_i = 6 - \frac{z_{-i}}{2}.$$

Since the equilibrium is symmetric, we obtain the contribution of each individual by solving the equation

$$z = 6 - \frac{2z}{2}.$$

that is,

$$\tilde{z} = 3$$

Hence, under voluntary contribution the state of knowledge is $K(\tilde{z}) = 9$, and the per capita consumption is

$$c^* = 9(12 - 3) = 81.$$

(c) (10 points) In the situation described in (b), study impact of a small and a large lump sum tax of T in units of time whose revenue is used to improve the technology. Is there a value of T that supports a Pareto optimal allocation of time.

Assume that the government imposes a lump sum tax $T \in [0, 12]$. Then each individual i chooses how much time he spends improving the technology z_i by solving the problem

$$\max_{z\in[0,12]} \left(z_i + z_{-i} + 3T\right)\left(12 - T - z_i\right),$$

where $z_{-i} = \sum_{j\neq i} z_j$. Hence

$$z_i = 6 - 2T - \frac{z_{-i}}{2}.$$

Since the equilibrium is symmetric, we obtain the contribution of each individual by solving the equation

$$z = 6 - 2T - \frac{2z}{2}.$$

The solution of this equation is

$$z = 3 - T.$$

Thus, the contribution of each individual is

$$\tilde{z}(T) = \max\{3 - T, 0\}.$$

If $T < 3$, then the tax has a null effect on the per capita consumption, i.e.,

$$c(T) = 81.$$

If $T > 3$, then no individual contributes (beyond the tax) to improving the technology, and the per capita consumption is

$$c(T) = 3T(12 - T) = 36T - 3T^2.$$

Thus, the optimal tax solves

$$36 - 6T = 0,$$

that is, $T^* = 6$, and $c(T^*) = c^*$. Hence the optimal tax supports the Pareto optimal allocation of time.

Exercise 3. The revenue of a risk-neutral principal is a random variable $X(e)$ taking values $x_1 = 0$ and $x_2 = 4$ with probabilities that depends on the level of effort of an agent, $e \in [0,1]$, and are given by $p_1(e) = 1 - e$ and $p_2(e) = e$, respectively. (Note that effort may take *any value* in the interval $[0,1]$.) The agent's preferences are represented by the von Neumann-Morgenstern utility function $u(w) = \sqrt{w}$, her reservation utility is $\underline{u} = 0$, and her costs of effort is $v(e) = e^2$.

(a) (5 points) Assume that *effort is verifiable*. Determine the optimal contract, and calculate the principal's profit and the social surplus.

Solution. Since $EX(e) = 4e$ *for* $e \in [0,1]$, *the principal's problem is*

$$\max_{(e,w)\in[0,1]\times\mathbb{R}_+} \quad 4e - w$$
$$s.t. \ \sqrt{w} \geq e^2.$$

Since the constrain is binding at a solution, this problem is equivalent to

$$\max_{e\in[0,1]} \ 4e - e^4.$$

An interior solution to this problem solves the equation

$$4 - 4e^3 = 1.$$

Hence the optimal contract is

$$(e^*, w^*) = (1, 1).$$

The principal profit is $EX(1) - 1 = 4 - 1 = 3$, *which is equals to the social surplus since the agent captures no surplus.*

(b) (10 points) Now assume that *effort not is verifiable.* Determine the optimal contract assuming that negative wages cannot be paid due to limited liability (because the agent has no income to give to the principle). Calculate the principal's profit and the lost of social surplus due to moral hazard.

Solution. Now the principal must take into account the incentives of the agent to exert effort given the wage schedule (w_1, w_2) *it offers. The agent expected utility of accepting the principal's offer is*

$$p_1(e)\sqrt{w_1} + p_2(e)\sqrt{w_2} - v(e) = (1-e)\sqrt{w_1} + e\sqrt{w_2} - e^2.$$

Hence when the agent accepts the contract he chooses the effort he exerts by solving the problem

$$\max_{e\in[0,1]} \; (1-e)\sqrt{w_1} + e\sqrt{w_2} - e^2.$$

Hence

$$\hat{e}(w_1, w_2) = \frac{\sqrt{w_2} - \sqrt{w_1}}{2}.$$

Note that effort decreases with $w_1 \geq 0$ *and increases with* $w_2 \geq 0$.

Since effort decreases with $w_1 \geq 0$, *it is optimal for the principal to set* $w_1 = 0$. *Denote* $w_2 = w$. *Therefore the principal must choose* w *in order to solve*

$$\max_{w\in\mathbb{R}_+} \; 4\hat{e}(0, w) - w = 2\sqrt{w} - w.$$

A solution to this problem satisfies the equation

$$\frac{1}{\sqrt{w}} = 1,$$

which implies a wage equal to 1. Hence the optimal contract is $(\hat{e}, \hat{w}) = (1, \hat{e}(0,1)) = (1, 1/2)$. *The profit to the principal is*

$$EX(1/2) - 1 = 2 - 1 = 1,$$

and the agent captures a surplus equal to

$$\hat{e}(0, \hat{w})\hat{w} - \hat{e}(0, \hat{w})^2 = \frac{1}{2}(1) - (1/2)^2 = 1/4.$$

In the following two questions assume that *effort is verifiable*, but that there are agents of two types, H and L, identical to the agent described above except for their cost of effort: the cost of the H type is the function given above, but the cost of the L type is twice as much.

(c) (5 points) Identify the contracts that the principal will offer to each type assuming that he *observes* the agent's type – use your results from part (a). Illustrate your findings providing a graph of the supply and demand schedules for agents of each type in the plane (e, w). Calculate the principals profit and the social surplus.

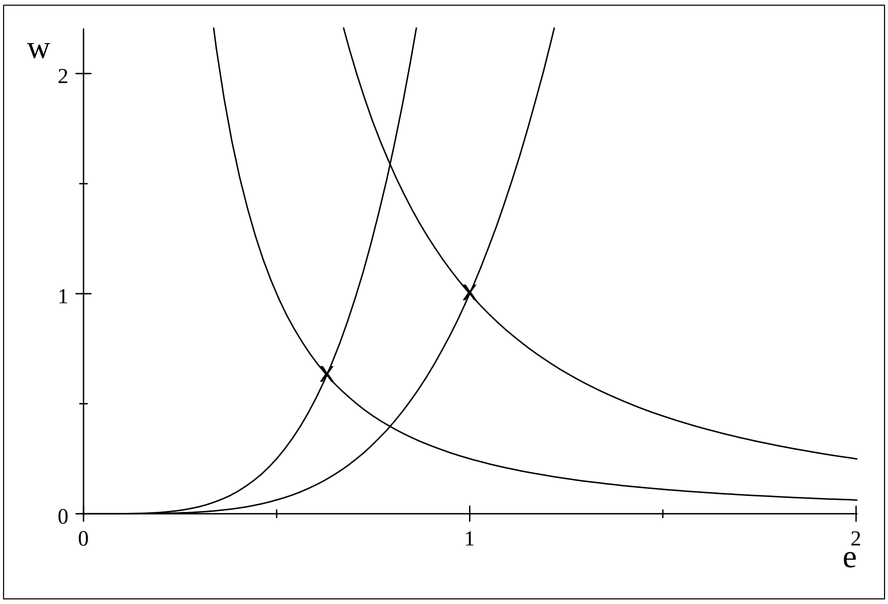
Solution. As calculated in part (a), the optimal contract for type H is $(e_H^*, w_H^*) = (1,1)$. The optimal contract for type L solves

$$\max_{(e,w)\in[0,1]\times\mathbb{R}_+} \quad 4e - w$$
$$s.t. \ \sqrt{w} \geq 2e^2.$$

That is,

$$\max_{e\in[0,1]} \ 4e - 4e^4,$$

which solution is $e_L^* = (1/4)^{1/3}$. Hence $(e_L^*, w_L^*) = ((1/4)^{1/3}, 2(1/4)^{2/3}) \simeq (0.63, 0.79)$.



The principal profit, which equals the social surplus, is

$$S^* = \frac{1}{2}\left(4(1) - 1\right) + \frac{1}{2}\left(4(\frac{1}{4})^{1/3} - 2(\frac{1}{4})^{2/3}\right) = 2.3631.$$

(d) (15 points) Now assume that the principal *does not observe* the agent's type, and that both types are present in equal measures in the population of agents. Assuming that he wants to offer a menu of contracts to hire both types of agents, write down the principal's problem, and the system of equations identifying the optimal menu. An approximate solution to this system of equations is $[(\tilde{e}_H, \tilde{w}_H), (\tilde{e}_L, \tilde{w}_L)] = [(0.92, 1.18), (0.489, 0.229)]$. Locate the optimal screening menu in the graph, and discuss the differences of this menu and that obtained in part (c). Calculate the principal's profit and verify that this menu is superior to offering a single contract. Calculate the lost in social surplus caused by adverse selection.

Solution. The principals problem is

$$\max_{[(e_H,w_H),(e_L,w_L)]\in([0,1]\times\mathbb{R}_+)^2} \frac{1}{2}(4e_H - w_H) + \frac{1}{2}(4e_L - w_L)$$

$$\begin{aligned} &s.t. \\ &\sqrt{w_L} \geq 2e_L^2 \qquad (PC_L) \\ &\sqrt{w_H} \geq e_H^2 \qquad (PC_H) \\ &\sqrt{w_L} - 2e_L^2 \geq \sqrt{w_H} - 2e_H^2 \qquad (IC_L) \\ &\sqrt{w_H} - e_H^2 \geq \sqrt{w_L} - e_L^2 \qquad (IC_H). \end{aligned}$$

The optimal menu of contracts is identified by system of equations:

$$\begin{aligned} 4 &= \frac{2e_H}{\frac{1}{2\sqrt{w_H}}} \\ 4 &= \frac{4e_L}{\frac{1}{2\sqrt{w_L}}} + \frac{\frac{1}{2}}{1-\frac{1}{2}}(2-1)\frac{2e_L}{\frac{1}{2\sqrt{w_H}}} \\ \sqrt{w_L} &= 2e_L^2 \qquad (IC_L) \\ \sqrt{w_H} - e_H^2 &= \sqrt{w_L} - e_L^2. \qquad (IC_H) \end{aligned}$$

The principal's profit with the optimal screening menu $[(\tilde{e}_H, \tilde{w}_H), (\tilde{e}_L, \tilde{w}_L)] = [(0.92, 1.18), (0.489, 0.22$ *is*

$$\frac{1}{2}(4(0.92) - 1.18) + \frac{1}{2}(4(0.489) - 0.229) = 2.1135.$$

The principal's profit if he offers only the optimal contract acceptable by type H agents, $(e_H^*, w_H^*) = (1, 1)$, *is*

$$\frac{1}{2}(4(1) - 1) = 1.5.$$

Hence the optimal screening menu is indeed optimal.

The surplus of the agents of type H is

$$\frac{1}{2}\left(w_H - e_H^2\right) = \frac{1}{2}\left(1.18 - (0.92)^2\right) = 0.1668.$$

Hence the social surplus

$$\tilde{S} = 2.1135 + 0.1668 = 2.2803 < 2.3631 = S^*.$$