5. In an exchange economy that extends over two periods, today and tomorrow, there are two consumers, A and B, and a single perishable good. The state of nature tomorrow can either be H or L. Consumers' preferences over consumption today and tomorrow are represented by the utility functions $u_A(x, y_H, y_L) = xy_H$, and $u_B(x, y_H, y_L) = xy_L$, and both consumers are endowed with four units of the good each date regardless of the state of nature. In this economy there are no markets for contingent contracts, but the are the spot markets as well as a credit market at date 1. Normalize the spot market prices to $p_0 = p_H = p_L = 1$, and denote by r the interest rate in the credit market.

(a) What will be the competitive equilibrium interest rate r^*? How much will each person borrow $b_i(r^*)$? Is the resulting allocation Pareto optimal?

CONSUMERS' BUDGET CONSTRAINTS:

$$x = 4 + b$$
$$y_H = 4 - (1+r)b$$
$$y_L = 4 - (1+r)b$$

CONSUMER $i \in \{A, B\}$ PROBLEM

$$\max_{b \in \mathbb{R}} \ (4+b)[4-(1+r)b]$$

HENCE $b^A(r) = b^B(r) := b(r)$.

MARKET CLEARING: $0 = b^A(r) + b^B(r) = 2b(r) \iff b(r) = 0$

HENCE CE ALLOCATION IS THE INITIAL ENDOWMENTS: $[(4,4,4),(4,4,4)]$

THE ALLOCATION $[(4,8,0),(4,0,8)]$ IS PARETO SUPERIOR.

(b) Now suppose there is also a security which for a price q pays one unit of consumption tomorrow if event H occurs. Determine the competitive equilibrium interest rate and security price, (r^*, q^*). Is the resulting allocation Pareto optimal? Check whether equilibrium involve either consumer *selling short*, i.e., selling more units of the security than the consumer is endowed with at state H.

PRICE NORMALIZATION : $p_x = p_H = p_L = 1$.

CONSUMERS' BUDGET CONSTRAINTS:

$$x = 4 + b - qz$$

$$y_H = 4 - (1+r)b + z$$

$$y_L = 4 - (1+r)b$$

CONSUMER A's PROBLEM:

$p_L > 0 \Rightarrow y_L^A = 0 \Rightarrow b^A(r,q) = \dfrac{4}{1+r}$,

AND z^A SOLVES :

$$\max_{z \in \mathbb{R}} \left(4 + \frac{4}{1+r} - qz\right)(4 - 4 + z)$$

$$z^A(r,q) = \frac{2(2+r)}{(1+r)q}$$

CONSUMER B's PROBLEM:

$p_H > 0 \Rightarrow y_H^B = 0 \Rightarrow b^B = \dfrac{4+z}{1+r}$

AND z^B SOLVES :

$$\max_{z \in \mathbb{R}} \left(4 + \frac{4+z}{1+r} - qz\right)[4 - (4+z)]$$

$$z^B(r,q) = \frac{2(2+r)}{(1+r)q - 1}$$

MARKET CLEARING:

CREDIT: $\frac{4}{1+r} + \frac{4 + \frac{2(2+r)}{(1+r)q-1}}{1+r} = 0$

SECURITY: $\frac{2(2+r)}{(1+r)q} + \frac{2(2+r)}{(1+r)q-1} = 0$

$r^* = 0$

$q^* = \frac{1}{2}$

CE ALLOCATION: $[(4,8,0), (4,0,8)]$.

(ELABORATE: HOW MUCH EACH CONSUMER BORROWS? HOW MANY UNITS OF THE SECURITY SHE BUYS? ANYBODY "SELLING SHORT"?)

Exercise 1. [redacted] Consider an economy that extends over two periods, today and tomorrow, in which there is a single perishable good, rice. In this economy there is a credit market which operates today, and there are spot markets operating tomorrow. There is a price-taking firm with a technology that for each unit of today's rice used as input produces 2 units of rice tomorrow if the state of nature is rainy, but only 1 unit if the state of nature is dry. Both states of nature are equally likely. The firm borrows rice today and supplies rice tomorrow with the objective of maximizing expected profits. There are two consumers whose preferences over consumption of rice today (x), tomorrow if rainy (y) and tomorrow if dry (z) are represented by the utility function $u(x, y, z) = x(y + z)$. Consumers share equally the property of the firm and each is endowed with 3 units of rice today. Calculate the competitive equilibrium prices and allocation.

Solution. If the firm borrows b units of rice today, then tomorrow it supplies inelastically 2b units of rice if the state is rainy and b units of rice if the state is dry. Hence b solves the problem

$$\max_{b \in \mathbb{R}_+} E[\pi(b)] := \frac{1}{2}(2b)\,p_y + \frac{1}{2}(b)\,p_z - (1+r)\,b = \left(p_y + \frac{p_z}{2} - (1+r)\right) b.$$

(The firm's expected profit is a linear function because the firm has constant returns to scale.) The firm's demand of credit is

$$b(p_y, p_z, r) = \begin{cases} 0 & \text{if } p_y + \dfrac{p_z}{2} < 1 + r \\ [0, \infty) & \text{if } p_y + \dfrac{p_z}{2} = 1 + r \\ \infty & \text{if } p_y + \dfrac{p_z}{2} > 1 + r. \end{cases}$$

Thus, in a CE $p_y + p_z/2 \leq 1 + r$*, and therefore the firm's expected profits are zero.*

Since the firm's profits are zero in a CE, each consumer faces the budget constraints

$$x = 3 - l,\ p_y y = (1+r)\,l,\ p_z z = (1+r)\,l,$$

where l are units of rice the consumer lends. A consumer's supply of credit l solves the problem

$$\max_{l \in \mathbb{R}} (3-l)\left(\frac{(1+r)\,l}{p_y} + \frac{(1+r)\,l}{p_z}\right) = \left(\frac{1+r}{p_y} + \frac{1+r}{p_z}\right)(3-l)\,l.$$

The solution to this problem is

$$3 - 2l = 0 \Leftrightarrow l(p_y, p_z, r) = \frac{3}{2};$$

that is, each consumer supplies 3/2 units of rice today as credit irrespective of the prices.

Market clearing in the credit market requires

$$b(p_y, p_z, r) = 2l(p_y, p_z, r) = 3 > 0,$$

which in turns requires

$$p_y + \frac{p_z}{2} = 1 + r. \tag{1}$$

Also, the market clearing conditions in the spot markets for rice tomorrow are

$$\begin{aligned} \frac{1+r}{p_y}\left(\frac{3}{2}\right) + \frac{1+r}{p_y}\left(\frac{3}{2}\right) &= 2\,(3) \Leftrightarrow \frac{1+r}{p_y} = 2 \\ \frac{1+r}{p_z}\left(\frac{3}{2}\right) + \frac{1+r}{p_z}\left(\frac{3}{2}\right) &= 3 \Leftrightarrow \frac{1+r}{p_z} = 1 \end{aligned}$$

Hence

$$1 + r = p_z = 2p_y. \tag{2}$$

The CE interest rate and prices must therefore satisfy equations (1) and (2). A particular CE equilibrium price vector is $(p_y^*, p_z^*, r^*) = (1, 2, 1)$. *The CE allocation is*

$$(x_1^*, y_1^*, z_1^*) = (x_2^*, y_2^*, z_2^*) = (\frac{3}{2}, 3, \frac{3}{2}).$$