Introduction to the Economics of Information: The Agency Problem

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The Agency Problem

$\triangleright$Two parties, which we refer to as *Principal* and *Agent*, bargain over the possibility of joint action.

$\triangleright$Join action generates a random revenue $X(t, e)$ which depends on the Agent's ability, i.e., type $t \in T$, and effort $e \in \mathbb{R}_+$.

$\triangleright$The conflict about the distribution of surplus between the Principal and the Agent, and the existence of asymmetric information about the Agent's type and/or the effort he exerts, are potential obstacles to reach an agreement that maximizes surplus.

The Agency Problem

$\triangleright$In order to avoid the distributional issue (i.e., the bargaining aspect of the problem), we assume that the Principal offers a *contract* (or a *menu of contracts)*, which the Agent either accepts or rejects, i.e., we give the entire bargaining power to the Principal.

$\triangleright$A contract is a pair (e, W), where $e \in \mathbb{R}_+$ is an effort request, and W a random wage promise.

The Agency Problem: Setting

- The *Principal's preferences* are represented by a twice differentiable Bernoulli utility function $\pi : \mathbb{R} \to \mathbb{R}$ such that $\pi(0) = 0$, $\pi' > 0$ and $\pi'' \leq 0$.

- The *Agent's preferences* are represented by a twice differentiable Bernoulli utility function $u : \mathbb{R} \to \mathbb{R}$ such that $u(0) = 0$, $u' > 0$ and $u'' \leq 0$.

- The Agent's *cost of effort* is given by a twice differentiable function $c : T \times \mathbb{R} \to \mathbb{R}$ such that $c(t, 0) = 0$, $\partial c / \partial e > 0$ and $\partial^2 c / \partial e^2 \geq 0$.

- The Agent's *reservation utility* is a real number $\underline{u} \geq 0$.

Note. Assuming that u and $\underline{u}$ are independent of the Agent's type is a simplification.

The Agency Problem: Timing

1. Nature selects the Agent's type $t \in T$.
2. The Principal proposes a menu of contracts $\{(e_t, W_t)\}_{t \in T}$.
3. The Agent rejects all contracts, or accepts one of the contracts and exerts effort.
4. Revenue and payoffs are realized.

The Agency Problem: Complete Information

When the Agent's ability is observable and effort is verifiable, Principal and Agent face a dynamic game of complete information.

In this game, the Principal, upon observing the Agent's type, offers a contract, (e, W). Here we focus on the two stages (sub)game identified by the Agent's type, which we omit.

In a subgame perfect equilibrium (SPE) of this game:

- The Agent rejects (accepts) any contract that gives him an expected utility less (greater) than his reservation utility.
- The Principal offers the contract that maximizes her expected utility.

The Agency Problem: Principal's Problem

Thus, in a SPE leading to an agreement, the contract offered by the Principal, (e^*, W^*), maximizes her welfare on the set of contracts that satisfy the Agent's *Participation Constrain (PC);* that is, (e^*, W^*) solves the problem:

$$\max_{(e,W)} \mathbb{E}[\pi(X(e) - W)]$$

subject to:

$$(PC) \qquad \mathbb{E}[u(W)] - c(e) \geq \underline{u}.$$

The Agency Problem: Simplifying Assumptions

Let us assume that:

- $X(e)$ is a discrete random variable with support $\{x_1, ..., x_n\}$ and density $p(e) = (p_1(e), ..., p_n(e))$, where $p_i(e) > 0$ for all $i \in \{1, ..., n\}$.
- Wage offers depend only on the realized revenue, but not on other random events unrelated to the activity, that is, $W = (w_1, ..., w_n) \in \mathbb{R}^n$.

Note. In some settings, e.g., under limited liability, contract offers must involved non-negative wages, i.e., $W \in \mathbb{R}^n_+$.

Taking as given (for now) the choice of effort e, the Lagrangian of the Principal's problem is

$$\mathcal{L}(e, w_1, ..., w_n, \lambda) = \sum_{i=1}^{n} p_i(e)\pi(x_i - w_i) + \lambda \left(\sum_{i=1}^{n} p_i(e)u(w_i) - c(e) - \underline{u} \right).$$

Taking derivative with respect to w_i we get

$$\frac{\partial \mathcal{L}}{\partial w_i} = -p_i(e)\pi'(x_i - w_i) + \lambda p_i(e)u'(w_i) = 0, \text{ for all } i \in \{1, ..., n\}.$$

Since $p_i(e) > 0$ for all $i \in \{1, ..., n\}$, and $u'(w_i) > 0$, then

$$\lambda = \frac{\pi'(x_i - w_i)}{u'(w_i)} > 0;$$

that is, the PC is binding. Hence, the FOC imply that the optimal wage contract satisfies:

$$\frac{\pi'(x_i - w_i)}{\pi'(x_j - w_j)} = \frac{u'(w_i)}{u'(w_j)},$$

which assures an optimal distribution of risk among Principal and Agent.

The Agency Problem: Optimal Wage Contract

(I) Assume that the Principal is risk neutral (i.e., $\pi(z) = \alpha z$, $\alpha \in \mathbb{R}_{++}$), and the Agent is risk averse (i.e., $u'' < 0$). Then

$$\frac{\pi'(x_i - w_i)}{\pi'(x_j - w_j)} = \frac{\alpha}{\alpha} = 1,$$

and since $u' > 0$, the FOC imply

$$\frac{u'(w_i)}{u'(w_j)} = 1 \Leftrightarrow u'(w_i) = u'(w_j) \Leftrightarrow w_i = w_j.$$

Hence the Principal offers a fix wage $\bar{w}$, i.e.,

$$W^* = (\bar{w}, ..., \bar{w}),$$

which is identified by the PC, i.e.,

$$u(\bar{w}) = c(e) + \underline{u} \Leftrightarrow \bar{w}(e) = u^{-1}\left(c(e) + \underline{u}\right).$$

The Agency Problem: Optimal Wage Contract

(II) Assume that the Principal is risk averse (i.e., $\pi'' < 0$), and the Agent is risk neutral (i.e., $u(w) = \beta w$, $\beta \in \mathbb{R}_{++}$). Then

$$\frac{u'(w_i)}{u'(w_j)} = \frac{\beta}{\beta} = 1,$$

and since $\pi' > 0$, the FOC imply

$$\begin{aligned} \frac{\pi'(x_i - w_i)}{\pi'(x_j - w_j)} = 1 \quad &\Leftrightarrow \quad \pi'(x_i - w_i) = \pi'(x_j - w_j) \\ &\Leftrightarrow \quad x_i - w_i = x_j - w_j := y \\ &\Leftrightarrow W^* = (x_1 - y, ..., x_n - y), \end{aligned}$$

where y (the Principal's profit) is identified by the PC,

$$\begin{aligned} \mathbb{E}[u(W)] \quad &= \quad \beta\mathbb{E}[W] = \beta\left(\mathbb{E}[X(e)] - y\right) = c(e) + \underline{u} \\ &\Leftrightarrow \quad y = \mathbb{E}[X(e)] - \frac{c(e) + \underline{u}}{\beta}. \end{aligned}$$

Thus, the optimal contract is a *franchise*!

The Agency Problem: Optimal Wage Contract

Example 1. Assume that there are two feasible levels of effort, $e \in \{0, 1\}$ with costs $c(e) = 5e$, and that X takes two values $x_1 = 160$ and $x_2 = 400$ with probabilities $p(e)$ and $1 - p(e)$, where $p(e) = (3 - 2e)/4$. Identify the optimal contract assuming that $\underline{u} = 1$, and

(a) $\pi(z) = z$ and $u(w) = \sqrt{w}$, and

(b) $\pi(z) = \sqrt{z}$ and $u(w) = w/20$.

(c) $\pi(z) = \sqrt{z}$ and $u(w) = \sqrt{w}$. (Hint. For $e \in \{0, 1\}$, solve the system

$$\begin{aligned} \frac{3-2e}{4}\sqrt{w_1} + \frac{1+2e}{4}\sqrt{w_2} &= 1 + 5e \\ \frac{\sqrt{400 - w_2}}{\sqrt{160 - w_1}} &= \frac{\sqrt{w_2}}{\sqrt{w_1}} \end{aligned}$$

and evaluate the Principal's expected utility at the solution.)

Example 1. (c)

$$w_1(0) = \frac{32}{169}\left(23 - 6\sqrt{10}\right), w_2(0) = \frac{80}{169}\left(23 - 6\sqrt{10}\right),$$
$$\mathbb{E}[\pi(X(0) - W^*(0))] \simeq 18.119$$

$$w_1\,(1) = \frac{1152}{1849}\left(47 - 6\sqrt{10}\right), w_2\,(1) = \frac{2880}{1849}\left(47 - 6\sqrt{10}\right)$$
$$\mathbb{E}[\pi(X(1) - W^*(1))] \simeq 17.143$$

The Agency Problem: Effort Choice

Let us assume that the set of feasible efforts is an interval $[0, \bar{e}]$, and for $e \in [0, \bar{e}]$ denote by $W^*(e)$ the Principal's optimal wage offer. The optimal contract involves an effort that solves the problem

$$\max_{e \in [0, \bar{e}]} \mathbb{E}[\pi(X(e) - W^*(e))].$$

(I) If the Principal is risk neutral, i.e., $\pi(z) = \alpha z$, where $\alpha > 0$, then as shown above $W^*(e) = (\bar{w}(e), ..., \bar{w}(e))$. Hence $\mathbb{E}[W^*(e)] = \bar{w}(e)$, and the Principal's problem becomes

$$\max_{e \in [0,\bar{e}]} \alpha \left(\mathbb{E}[(X(e)] - \bar{w}(e) \right),$$

which amounts to maximizing expected profit. An interior solution solves the equation

$$\mathbb{E}'[(X(e)] = \bar{w}'(e),$$

i.e., marginal revenue equals marginal cost.
Since $\bar{w}(e)$ solves the equation $u(w) = c(e) + \underline{u}$, differentiating we get

$$u' dw = c' de,$$

that is

$$\frac{dw}{de} = \bar{w}'(e) = \frac{c'}{u'}.$$

The Agency Problem: Effort Choice

Substituting in the FOC identifying the solution to Principal's problem we get the equation

$$\mathbb{E}'[(X(e)] = \frac{c'(e)}{u'(w)},$$

which defines the Principal's demand of effort, $D(w)$. Likewise, the Agent's participation constraint

$$u(w) = c(e) + \underline{u},$$

defines the Agent's supply of effort, $S(w)$. The optimal effort e^* solves the equation

$$S(w) = D(w).$$

And the optimal contract is (e^*, W^*), where $W^* = (\bar{w}(e^*), ..., \bar{w}(e^*))$.

The Agency Problem: Effort Choice

Note that the second order sufficient condition for such a contract to be a solution,

$$\mathbb{E}''[(X(e)] - \bar{w}''(e) < 0,$$

is satisfied whenever $\mathbb{E}[(X(e)]$ is a concave function of effort, that is, $\mathbb{E}''[(X(e)] < 0$, since

$$\bar{w}''(e) = \frac{d}{de}\left(\frac{c'(e)}{u'(\bar{w}(e))}\right) = \frac{c''u' - c'u''\bar{w}'}{(u')^2} > 0.$$

(This inequality implies that the Principal's demand of effort is decreasing. Likewise, it can be shown that the Agent's supply of effort is increasing.)

(II) If the Principal is risk averse and the Agent is risk neutral, then as shown above for each level of effort the optimal wage leads to the constant profit $y(e) = \mathbb{E}[(X(e)] - (c(e) + \underline{u}) / \beta$. Since π is increasing, then maximizing $\pi(y(e))$ amounts to maximizing $y(e)$. Hence the Principal's problem becomes

$$\max_{e \in [0, \bar{e}]} \mathbb{E}[(X(e)] - \frac{c(e) + \underline{u}}{\beta}.$$

An interior solution solves the equation

$$\mathbb{E}'[(X(e)] = \frac{c'(e)}{\beta}.$$

The second order sufficient condition is

$$\mathbb{E}''[(X(e)] - \frac{c''(e)}{\beta} < 0,$$

which is satisfied whenever $\mathbb{E}[(X(e)]$ is a concave function of effort.

The Agency Problem: Effort Choice

Example 2. Assume that the set of feasible effort levels is $[0,1]$, and that the cost of effort is $c(e) = e/2$. Revenue X takes two values $x_1 = 4$, and $x_2 = 8$ with probabilities $p(e)$ and $1 - p(e)$, where $p(e) = (2 - e)/3$. Identify the optimal contract assuming that $\underline{u} = 1$, and

(a) $\pi(z) = z$ and $u(w) = \sqrt{w}$, and

(b) $\pi(z) = \sqrt{z}$ and $u(w) = \beta w$.

(c) $\pi(z) = \sqrt{z}$ and $u(w) = \sqrt{w}$.

SOLUTION

$$\mathbb{E}[X(e)] = p(e)\, x_1 + (1-p(e))\, x_2 \iff \boxed{\mathbb{E}[X(e)] = \frac{1}{3}(16+4e)}$$

(a) THE OPTIMAL WAGE CONTRACT IS, IN THIS CASE, A FIX WAGE $\bar{w}(e)$, WHERE $\bar{w}(e)$ SOLVES THE PC EQUATION

$$u(w) = c(e) + \underline{u} \iff \sqrt{w} = \frac{e}{2} + 1 \iff \boxed{\bar{w}(e) = \left(\frac{e}{2}+1\right)^2}$$

HENCE, THE OPTIMAL EFFORT SOLVES

$$\max_{e\in[0,1]} \frac{1}{3}(16+4e) - \left(\frac{e}{2}+1\right)^2$$

THAT IS, e^* SOLVES

$$\frac{4}{3} - \frac{e}{2} - 1 = 0 \iff \boxed{e^* = \frac{2}{3}} \Rightarrow \bar{w}(e^*) =$$

THE OPTIMAL CONTRACT IS $\underline{(e^*, w^*) = \left(\frac{2}{3}, \left(\frac{16}{9}, \frac{16}{9}\right)\right)}$.

(b) In this case the optimal wage contract is a franchese, and the level of effor solves the problem

$$\max_{e\in[0,1]} y(e) = \mathbb{E}\left[X(e)\right] - \frac{C(e)+u}{\beta} = \frac{1}{3}(16+4e) - \frac{\left(\frac{e}{2}+1\right)}{\beta}.$$

$$= \left(\frac{16}{3} - \frac{1}{\beta}\right) + \left(\frac{4}{3} - \frac{1}{2\beta}\right)e$$

Hence

$$(e^*, y(e^*)) = \begin{cases} \left(0, \frac{16}{3} - \frac{1}{\beta}\right) & \text{if } \beta < \frac{3}{8} \\ \left(e, \frac{8}{3}\right) \ \forall e \in [0,1] & \text{if } \beta = \frac{3}{8} \\ \left(1, \frac{20}{3} - \frac{3}{2\beta}\right) & \text{if } \beta > \frac{3}{8}. \end{cases}$$

(c) THE OPTIMAL WAGE CONTRACT $W=(w_1, w_2)$ IS THE SOLUTION TO THE SYSTEM

$$\left.\begin{array}{lll} Eu(W) = c(e) + \underline{u} & \Longleftrightarrow & p(e)\sqrt{w_1} + (1-p(e))\sqrt{w_2} = \frac{e}{2} + 1 \\ \dfrac{\pi'(x_1 - w_1)}{\pi'(x_2 - w_2)} = \dfrac{u'(w_1)}{u'(w_2)} & \Longleftrightarrow & \dfrac{x_2 - w_2}{x_1 - w_2} = \dfrac{\sqrt{w_1}}{\sqrt{w_2}} \end{array}\right\}$$

<u>SOLUTION</u>

$$w_1(e) = 4 + g(e), \quad w_2(e) = 8 + 2g(e),$$

WHERE

$$g(e) = \frac{(18\sqrt{2}+11)e^4 - (148-24\sqrt{2})e^3 - (798+36\sqrt{2})e^2 + (798-216\sqrt{2})e + 152 - 144\sqrt{2}}{(2e^2 + 16e - 4)^2}$$

HENCE

$$E\pi(X(e) - W(e)) = p(e)\sqrt{4 - w_1(e)} + (1-p(e))\sqrt{8 - w_2(e)}$$

$$= p(e)\sqrt{g(e)} + (1-p(e))\sqrt{2g(e)}$$

$$= [p(e) + \sqrt{2}(1-p(e)]\sqrt{g(e)}$$

A NUMERICAL SOLUTION TO THE EQUATION

$$\frac{d}{de}\left\{\left[p(e)+\sqrt{2}\,(1-p(e))\right]\sqrt{g/e}\right\}=0$$

is

$$e^{*}=0.74,\quad w_1(e^{*})=4-2.78,\quad w_2(e^{*})=8-2/2.78).$$

The Agency Problem: Moral Hazard

Assume that effort is *not verifiable,* i.e., it cannot be proven in court, and hence is *not contractible*, that is, the contract the Principal offers the Agent cannot be made contingent on effort. (Whether or not effor is observed by the Principal is irrelevant.)

The timing of the game is now:

1. Nature selects the Agent's type $t \in T$.
2. The Principal proposes a menu of contracts $\{W_t\}_{t \in T}$.
3. The Agent rejects all contracts, or accepts one of the contracts and *chooses* his effort $e \in [0, \bar{e}]$.
4. Revenue and payoffs are realized.

The Agency Problem: Moral Hazard

Let us assume that the Agent's type is observed by the Principal.

In a subgame perfect equilibrium of this game, the Principal offers a random wage W that solves the problem

$$\max_{W} \mathbb{E}[\pi(X(e^*(W)) - W)]$$

subject to:

$$(PC) \qquad \mathbb{E}[u(W)] - c(e^*(W)) \geq \underline{u},$$

where

$$e^*(W) := \arg \max_{e \in [0,\bar{e}]} \mathbb{E}[u(W(e)] - c(e).$$

The Agency Problem: Moral Hazard

We maintain our simplifying assumptions above,

- $X(e)$ is a discrete random variable with support $\{x_1, ..., x_n\}$ and distribution $\{p_1(e), ..., p_n(e)\}$,
- The wage offer can be conditioned only on the realized revenue, but not on other random events unrelated to the activity, that is, $W = (w_1, ..., w_n) \in \mathbb{R}^n$. (Again, under limited liability wage offers are further restricted, e.g., $W \in \mathbb{R}^n_+$.)

In addition, we assume that

- The Principal is risk neutral and the Agent is risk averse, and
- the set of feasible efforts is $\{e_l, e_h\}$, with $e_l < e_h$.

The contract $(e_l; \bar{w}(e_l))$, where $u(\bar{w}(e_l)) = c(e_l) + \underline{u}$, satisfies both the participation and incentive compatibility constraints. Therefore this is the contract involving low effort that maximizes expected profit.

As for the expected profit maximizing contract involving high effort, it must be a solution to problem P_h given by

$$\max_{W} \mathbb{E}[X(e_h) - W(e_h)]$$

subject to:

$$(PC) \qquad \mathbb{E}[u(W)] - c(e_h) \geq \underline{u}$$

$$(IC) \qquad \mathbb{E}[u(W(e_h)] - c(e_h) \geq \mathbb{E}[u(W(e_l)] - c(e_l).$$

The optimal contract is that leading to the largest expected profit among these two contracts.

The Agency Problem: Moral Hazard

The Lagrangian of the problem P_h is:

$$\begin{aligned}\mathcal{L}(w_1, ..., w_n, \lambda, \mu) &= \sum_{i=1}^{n} p_i(e_h)(x_i - w_i) \\ &+\lambda \left(\sum_{i=1}^{n} p_i(e_h)u(w_i) - c(e_h) - \underline{u} \right) \\ &+\mu \left(\sum_{i=1}^{n} p_i(e_h)u(w_i) - c(e_h) \right. \\ &\qquad \left. - \left(\sum_{i=1}^{n} p_i(e_l)u(w_i) - c(e_l) \right) \right).\end{aligned}$$

The Agency Problem: Moral Hazard

The optimal wage contract must satisfy for all i:

$$\frac{\partial \mathcal{L}}{\partial w_i} = -p_i(e_h) + \lambda p_i(e_h) u'(w_i) + \mu u'(w_i)(p_i(e_h) - p_i(e_l)) = 0.$$

Denoting $l_i = p_i(e_l)/p_i(e_h)$, this equation may be written as

$$\frac{1}{u'(w_i)} = \lambda + \mu(1 - l_i).$$

This equation has a clear interpretation. Since

$$\mathbb{E}\left[\frac{1}{u'(W)}\right] = \sum_{i=1}^{n} \frac{p_i(e_h)}{u'(w_i)} = \lambda,$$

if $l_i < 1$, then w_i is above the average wage, and vice versa; i.e., providing incentives to exert high effort requires paying larger wages if the realized revenue is more likely when the effort is high than when it is low.

The Agency Problem: Moral Hazard

The participation constraint is binding since

$$0 < \mathbb{E}\left[\frac{1}{u'(W)}\right] = \lambda.$$

Further, a fixed wage contract $\bar{w} = \bar{w}(e_h)$, which is optimal in the absence of the IC constraint, is not feasible since

$$\begin{aligned} \sum_{i=1}^{n} p_i(e_h)u(\bar{w}) - c(e_h) &= u(\bar{w}) - c(e_h) \\ &< u(\bar{w}) - c(e_l) \\ &= \sum_{i=1}^{n} p_i(e_l)u(\bar{w}) - c(e_l). \end{aligned}$$

Thus, the IC constraint is binding, i.e., $\mu > 0$, and hence the Principal's payoff and the total surplus, which coincide, are lower with moral hazard.

The Agency Problem: Moral Hazard

Example 3. Consider an agency problem in which $\pi(z) = z$, $u(w) = \sqrt{w}$, $c(e) = e$, and $\underline{u} = 1$. Effort may be either $e = 0$ or $e = 1$, and it is not verifiable. Revenue X takes two values $x_1 = 4$, and $x_2 = 8$ with probabilities $p(e) = (2 - e)/3$ and $1 - p(e) = (1 + e)/3$, respectively. Identify the optimal contract. (Hint: $w_1 = 0, w_2 = 9$.)

Example 3'. Consider an agency problem in which $\pi(z) = z$, $u(w) = \sqrt{w}$, $c(e) = e$, and $\underline{u} = 1$. Effort may be either $e = 0$ or $e = 1$, and it is not verifiable. Revenue X takes three values $x_1 = 4$, $x_2 = 8$ and $x_3 = 18$ with probabilities $p_1(e) = (2 - e)/3$ and $p_2(e) = 1/3$, respectively. Identify the optimal contract. (Hint: $w_1 = 1/4$, $w_2 = 4$, $w_3 = 49/4$, $\lambda = 4$, $\mu = 3$.)

The Agency Problem: Moral Hazard

Assume that the set of feasible efforts is a bounded interval $[0, \bar{e}]$. Under our assumptions about the functions u and c, facing an acceptable wage offer the Agent solves the convex problem

$$\max_{e \in [0,\bar{e}]} \mathbb{E}[u(W(e)] - c(e).$$

This problem has a solution, and if it is unique it is identified by the first order condition

$$\mathbb{E}'[u(W(e)] = c'(e).$$

In this is the case we can treat effort as a choice variable in the Principal's problem, while incorporating this differential equation as a constraint. (This method is known as the *first order approach*.)

Principal's problem with moral hazard:

$$\max_{(e,W)\in[0,\bar{e}]\times\mathbb{R}^n} \mathbb{E}[X(e)] - \mathbb{E}[W(e)]$$

subject to:

$$(PC) \qquad \mathbb{E}[u(W(e))] - c(e) \geq \underline{u}$$

$$(IC) \qquad \mathbb{E}'[u(W(e))] = c'(e).$$

The Lagrangian of this problem is:

$$\begin{aligned}\mathcal{L}(e, W, \lambda, \mu) &= \mathbb{E}[X(e)] - \mathbb{E}[W(e)] \\ &+\lambda\left(\mathbb{E}[u(W(e))] - c(e) - \underline{u}\right) \\ &+\mu\left(\mathbb{E}'[u(W(e))] - c'(e)\right).\end{aligned}$$

The Agency Problem: Moral Hazard

Taking derivative with respect to effort we get

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial e} &= \mathbb{E}'[X(e)] - \mathbb{E}'[W(e)] \\
&\quad + \lambda \left(\mathbb{E}'[u(W(e))] - c'(e) \right) \\
&\quad + \mu \left(\mathbb{E}''[u(W(e))] - c''(e) \right).
\end{aligned}$$

Hence optimal effort satisfies the equation

$$\mathbb{E}'[X(e)] = \mathbb{E}'[W(e)] + \mu \left(\mathbb{E}''[u(W(e))] - c''(e) \right).$$

In this equation the RHS is the marginal cost of increasing effort, which involves two terms associated to the participation and incentive constraints, respectively.

Example 4. Consider an agency problem in which $\pi(z) = z$, $u(w) = \sqrt{w}$, $c(e) = e^2/2$, and $\underline{u} = 1$. The set of feasible effort levels e is $[0, 1]$. Effort is not verifiable. Revenue X takes two values $x_1 = 4$, and $x_2 = 8$ with probabilities $p(e) = (2 - e)/3$ and $1 - p(e) = (1 + e)/3$, respectively. Identify the optimal contract.

The Agency Problem: Moral Hazard

In the absence of moral hazard,

$$\mathbb{E}[X(e)] - \bar{w}(e) = \frac{4}{3}e + \frac{16}{3} - \left(1 + \frac{e^2}{2}\right)^2$$

and

$$\frac{d}{de}\left(\mathbb{E}[X(e)] - \bar{w}(e)\right) = -e^3 - 2e + \frac{4}{3} = 0 \Leftrightarrow \hat{e} = 0.57273.$$

Hence

$$\mathbb{E}[X(\hat{e})] - \bar{w}(\hat{e}) = 4.7421$$

With moral hazard the lagrangian of the Principal'sproblem is

$$\begin{aligned}\mathcal{L}(w_1, w_2, e, \lambda, \mu) &= \frac{2-e}{3}(4 - w_1) + \frac{1+e}{3}(8 - w_2) \\ &+\lambda\left(\frac{2-e}{3}\sqrt{w_1} + \frac{1+e}{3}\sqrt{w_2} - \frac{e^2}{2} - 1\right) \\ &+\mu\left(-\frac{\sqrt{w_1}}{3} + \frac{\sqrt{w_2}}{3} - e\right).\end{aligned}$$

The Agency Problem: Moral Hazard

A (numerical) solution to the system

$$\frac{\partial \mathcal{L}}{\partial w_1} = 0, \frac{\partial \mathcal{L}}{\partial w_2} = 0, \frac{\partial \mathcal{L}}{\partial e} = 0, \frac{\partial \mathcal{L}}{\partial \lambda} = 0, \frac{\partial \mathcal{L}}{\partial \mu} = 0$$

is

$$w_1^* = 0.59821, \;\; w_2^* = 1.9315, \;\; e^* = 0.20545, \;\; \lambda^* = 2.0422, \;\; \mu^* = 0.88889.$$

The Principal's expected profit is

$$\begin{aligned} \mathbb{E}[X(e^*)] - W^*(e^*) &= \frac{2 - 0.20545}{3}(4 - 0.59821) \\ &\quad + \frac{1 + 0.20545}{3}(8 - 1.9315) \\ &= 4.4733. \end{aligned}$$

The surplus loss due to moral hazard is

$$(\mathbb{E}[X(\hat{e})] - \bar{w}(\hat{e})) - (\mathbb{E}[X(e^*)] - W^*(e^*)) = 4.7421 - 4.4733 = 0.2688.$$

The Agency Problem: Adverse Selection (Screening)

Let us now consider an agency problem in which the risk neutral Principal faces a heterogenous population of risk averse agents.

In this population, all agents have the same utility function u and reservation utility $\underline{u}$, but their costs of effort differ: for a fraction $q \in (0, 1)$ of agents (type H) the cost of effort is $kc(e)$, where $k > 1$, while for the remaining fraction $1 - q$ (type L), the cost of effort is $c(e)$.

The Agent is randomly drawn from this population, and his type, H or L, is *not* observed by the Principal.

In order to simplify the problem we assume that effort is verifiable.

The Agency Problem: Adverse Selection (Screening)

In this setting, Principal and Agent face a four stages game of asymmetric information:

1. Nature selects the Agent's type $t \in \{H, L\}$ with probabilities q and $1 - q$. The Agent's type is NOT observed by the Principal.
2. The Principal offers the Agent a contract (e, w) or a menu of contracts $\{(e_H, w_H), (e_L, w_L)\}$.
3. The Agent either accepts one of the contracts and exerts effort, or rejects all contracts.
4. Revenue and payoffs are realized.

In a Perfect Bayesian equilibrium (PBE) of this game, the Principal offers a menu of contracts that maximizes his expected utility, anticipating that each type of agent will respond optimally.

The Agency Problem: Adverse Selection (Screening)

The Principal may offer either:

- The single contract, $(e^*, \bar{w}(e^*)) \in \mathbb{R}^2_+$, where $\bar{w}(e) = u^{-1}(c(e) + \underline{u})$ and $e^* = \arg\max_e \mathbb{E}[X(e)] - \bar{w}(e)$, which only type L agents accept.

Or:

- A menu of contracts, $\{(e_H, w_H), (e_L, w_L)\} \in \mathbb{R}^4_+$, designed in such a way that agents of type H accept the contract (e_H, w_H) and agents of type L accept the contract (e_L, w_L).

Note that the menu includes the possibility of offering the same contract to both types.

The Agency Problem: Adverse Selection (Screening)

In a PBE in which the Principal offers a menu of contracts, such menu solves the problem:

$$\max_{\{(e_H,w_H),(e_L,w_L)\}\in\mathbb{R}^4_+} q\left(\mathbb{E}[X(e_H)] - w_H\right) + (1-q)\left(\mathbb{E}[X(e_L)] - w_L\right)$$

subject to:

$$\begin{array}{ll}
(PC_H) & u(w_H) \geq kc(e_H) + \underline{u} \\
(PC_L) & u(w_L) \geq c(e_L) + \underline{u} \\
(IC_H) & u(w_H) - kc(e_H) \geq u(w_L) - kc(e_L) \\
(IC_L) & u(w_L) - c(e_L) \geq u(w_H) - c(e_H).
\end{array}$$

The Agency Problem: Adverse Selection (Screening)

This problem may be simplified by noticing that in this problem PC_L is implied by IC_L and PC_H, since

$$\begin{aligned} u(w_L) - c(e_L) &\geq u(w_H) - c(e_H) \text{ (by } IC_L) \\ &\geq u(w_H) - kc(e_H) \text{ (because } k > 1) \\ &\geq \underline{u} \text{ (by } PC_H). \end{aligned}$$

(The second inequality is strict if $e_H > 0$. In this case PC_L is non-binding, and the low type captures some rents.)

Hence PC_L can be ignored.

Suppressing the inequality PC_L we may write the Lagrangian as:

$$\begin{aligned}\mathcal{L}(\cdot) \quad = \quad & q\left(\mathbb{E}[X(e_H)] - w_H\right) + (1-q)\left(\mathbb{E}[X(e_L)] - w_L\right) \\ & + \lambda_H\left(u(w_H) - kc(e_H) - \underline{u}\right) \\ & + \mu_H\left(u(w_H) - kc(e_H) - u(w_L) + kc(e_L)\right) \\ & + \mu_L\left(u(w_L) - c(e_L) - u(w_H) + c(e_H)\right).\end{aligned}$$

The Agency Problem: Adverse Selection (Screening)

The first order conditions identifying an interior solution of the problem are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial e_H} &= q\mathbb{E}'[X(e_H)] - \lambda_H kc'(e_H) - \mu_H kc'(e_H) + \mu_L c'(e_H) = 0 \\
\frac{\partial \mathcal{L}}{\partial w_H} &= -q + \lambda_H u'(w_H) + \mu_H u'(w_H) - \mu_L u'(w_H) = 0 \\
\frac{\partial \mathcal{L}}{\partial e_L} &= (1-q)\mathbb{E}'[X(e_L)] + \mu_H kc'(e_L) - \mu_L c'(e_L) = 0 \\
\frac{\partial \mathcal{L}}{\partial w_L} &= -(1-q) - \mu_H u'(w_L) + \mu_L u'(w_L) = 0,
\end{aligned}$$

This system may be rewritten as

$$
\begin{aligned}
q\frac{\mathbb{E}'[X(e_H)]}{c'(e_H)} &= \lambda_H k - (\mu_L - \mu_H k) && (1)\\
\frac{q}{u'(w_H)} &= \lambda_H - (\mu_L - \mu_H) && (2)\\
(1-q)\frac{\mathbb{E}'[X(e_L)]}{c'(e_L)} &= \mu_L - k\mu_H && (3)\\
\frac{1-q}{u'(w_L)} &= \mu_L - \mu_H. && (4)
\end{aligned}
$$

The Agency Problem: Adverse Selection (Screening)

In addition, the complementary slackness conditions

$$\lambda_H \left(u(w_H) - kc(e_H) - \underline{u}\right) = 0 \quad (5)$$
$$\mu_H \left(u(w_H) - kc(e_H) - u(w_L) + kc(e_L)\right) = 0 \quad (6)$$
$$\mu_L \left(u(w_L) - c(e_L) - u(w_H) + c(e_H)\right) = 0 \quad (7)$$

must hold.

Since $\mu_L > 0$ by equation (4), then equation (7) implies that IC_L is binding.

Since $\lambda_H > 0$ by equations (2) and (4), then equation (5) implies that PC_H is binding.

We show that in a solution to this system IC_H is non-binding, that is

$$u(w_H) - kc(e_H) > u(w_L) - kc(e_L),$$

and therefore $\mu_H = 0$ by equation (7).

We first show that $e_L \geq e_H$. By IC_L and $\mu_L > 0$,

$$\begin{aligned} c(e_L) - c(e_H) &= u(w_L) - u(w_H) \\ \text{(by } IC_H\text{)} &\leq k\left(c(e_L) - c(e_H)\right), \end{aligned}$$

which implies

$$(1-k)\left(c(e_L) - c(e_H)\right) \leq 0 \Leftrightarrow c(e_L) \geq c(e_H) \Leftrightarrow e_L \geq e_H.$$

Next we show that $e_L \neq e_H$.

Suppose by way of contradiction that $e_L = e_H$. This implies that $w_L = w_H$ for otherwise both types will choose the contract involving the largest wage (for identical effort).

Formally, the inequality above implies

$$0 = c(e_L) - c(e_H) = u(w_L) - u(w_H).$$

Hence

$$u(w_L) - u(w_H) = 0 \Leftrightarrow w_L = w_H.$$

But if $e_L = e_H$ and $w_L = w_H$, then we can suppress the arguments in the system of first order conditions, and write it as:

$$
\begin{aligned}
q\frac{\mathbb{E}'[X]}{c'} &= \lambda_H k - (\mu_L - \mu_H k) \qquad (1)\\
\frac{q}{u'} &= \lambda_H - (\mu_L - \mu_H) \qquad (2)\\
(1-q)\frac{\mathbb{E}'[X]}{c'} &= \mu_L - k\mu_H \qquad (3)\\
\frac{1-q}{u'} &= \mu_L - \mu_H. \qquad (4)
\end{aligned}
$$

Substituting $\lambda_H = 1/u'$ from (2) and (4) into equation (2) we get $(1-q)\lambda_H = \mu_L - \mu_H$

$$\mu_L = (1-q)\lambda_H + \mu_H.$$

Substituting $k\lambda_H = (\mathbb{E}X)' / c'$ from (1) and (3) into equation (1) we get

$$\mu_L = k[(1-q)\lambda_H + \mu_H].$$

Since $k > 1$ and $(1-q)\lambda_H + \mu_H > 0$ these two equations cannot hold. Hence

$$e_L \neq e_H,$$

and since $e_L \geq e_H$, we have

$$e_L > e_H.$$

Finally we show that IC_H holds with strict inequality, and therefore $\mu_H = 0$ by the complementary slackness condition (6).

Since $e_L > e_H$ implies $c(e_L) - c(e_H) > 0$, and IC_L is binding (because $\mu_L > 0$), then by IC_L

$$u(w_L) - u(w_H) = c(e_L) - c(e_H) < k(c(e_L) - c(e_H)),$$

where the inequality follows since $k > 1$. Hence

$$u(w_L) - kc(e_L) < u(w_H) - kc(e_H).$$

Therefore IC_H is non-binding, and $\mu_H = 0$.

Substituting $\mu_H = 0$ into the system of first order conditions, we get

$$q\frac{\mathbb{E}'[X(e_H)]}{c'(e_H)} = \lambda_H k - \mu_L \qquad (1)$$

$$\frac{q}{u'(w_H)} = \lambda_H - \mu_L \qquad (2)$$

$$(1-q)\frac{\mathbb{E}'[X(e_L)]}{c'(e_L)} = \mu_L \qquad (3)$$

$$\frac{1-q}{u'(w_L)} = \mu_L. \qquad (4)$$

This system may be rewritten as

$$\mathbb{E}'[X(e_H)] = \frac{kc'(e_H)}{u'(w_H)} + \frac{1-q}{q}(k-1)\frac{c'(e_H)}{u'(w_L)} \qquad (1,2)$$

$$\mathbb{E}'[X(e_L)] = \frac{c'(e_L)}{u'(w_L)} \qquad (3,4).$$

These two equations together with the two binding constrains

$$u(w_H) = kc(e_H) + \underline{u} \qquad (5)$$

$$u(w_L) - c(e_L) = u(w_H) - c(e_H) \qquad (7)$$

identify the optimal contract.

The Agency Problem: Adverse Selection (Screening)

Properties of the optimal menu:

- The contract offered to the low cost type is *optimal:* by equation $(3,4)$, the Principal selects a contract on her demand of effort for the low cost type.
- The contract offered to the high cost type involves less effort than the optimal contract with complete information, i.e., the contract satisfying equation (1,2) is below the Principal's demand of effort for the high cost type. Reducing effort (and correspondingly wage) makes the contract for the high cost type less attractive to the low cost type, which relaxes the incentive constraint for this type. Hence adverse selection distorts downwards the demand of effort of the high cost type.
- As observed earlier, the low cost type captures a positive surplus – which we can refer to as *information rents*.

Exercise. $\mathbb{E}[X(e)] = 2e$, and $u(x) = x$, $c(e) = e^2$, $\underline{u} = 0$, $k = 2$, $q = 1/2$.

Optimal contracts with complete information:

Effort Supplies: $w_H = 2e_H^2$; $w_L = e_L^2$.
Effort Demands: $2 = 4e_H$, i.e., $e_H = 1/2$; $2 = 2e_L$, i.e., $e_L = 1$.

Thus, the optimal contracts are

$$(e_L^*, w_L^*) = (1, 1), (e_H^*, w_H^*) = (1/2, 1/2).$$

And the principal's expected profit is

$$\mathbb{E}[\pi^*] = \frac{1}{2}(2(1) - 1) + \frac{1}{2}\left(2\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right)^2\right) = \frac{3}{4},$$

which is equal to the total surplus.

The Agency Problem: Adverse Selection (Screening)

With adverse selection the optimal menu of contracts solves,

$$\begin{aligned} 2 &= 2e_L \\ 2 &= 2\frac{2e_H}{1} + \frac{1-\frac{1}{2}}{\frac{1}{2}}(2-1)\frac{2e_H}{1} \\ w_H &= 2e_H^2 \\ w_L - e_L^2 &= w_H - e_H^2 \end{aligned}$$

Solving the system we get

$$(\tilde{e}_L, \tilde{w}_L) = (1, 10/9), (\tilde{e}_H, \tilde{w}_H) = (1/3, 2/9).$$

The expected profit is

$$\mathbb{E}[\tilde{\pi}] = \frac{1}{2}\left(2(1) - \frac{10}{9}\right) + \frac{1}{2}\left(2\left(\frac{1}{3}\right) - \frac{2}{9}\right) = \frac{2}{3} < \mathbb{E}[\pi^*] = \frac{3}{4}.$$

Hence Adverse selection reduces the profits of the Principal!

The information rents captured by the low types are

$$\frac{1}{2}\left(\frac{1}{9}\right) = \frac{1}{18}.$$

Hence the total surplus with adverse selection is

$$S = \frac{2}{3} + \frac{1}{18} = \frac{3}{4} - \frac{1}{36}.$$

Adverse selection reduces the social surplus!

The Agency Problem: Signaling

(A simple version of M. Spence – Job Market Signaling, QJE 1973.)

Consider a competitive labor market in which a fraction $q \in (0, 1)$ are low skilled workers (type L), while for the remaining fraction $1 - q$ are high skilled workers (type H).

We assume that there is no moral hazard, and that the joint action with either type of worker generates a positive surplus. Specifically, the revenues and the agents' reservation utilities satisfy

$$x_H > u_H > x_L > u_L,$$

If workers' types are observable, then there will be separated labor markets for each type of workers. Moreover, because principals have constant returns to scale, in a competitive equilibrium their profits are nil, and therefore the workers' market wage are

$$w_H^* = x_H, \; w_L^* = x_L,$$

and all workers supply their labor.

The Agency Problem: Signaling

If a worker's types is private information, then there will be a single labor market in which all workers participate.

In a CE of this market, the principals' profits are zero as well.

The market has a CE in which the wage is

$$w^* = x_L,$$

and only low skilled workers supply labor.

Moreover, if

$$\bar{x}(q) := qx_L + (1-q)\,x_H > u_H,$$

then there is also a (pooling) CE, in which the wage is

$$w^* = \bar{x}(q),$$

and both types of workers supply labor.

The Agency Problem: Signaling

Let us assume instead that principals and workers interact bilaterally in random encounters, and that before a meeting occurs, each agent may try to *signal* his type by taking a costly action $y \in [0, \bar{y}]$; e.g., completing a university degree, running a marathon, going to the beauty parlor (or to the gym), getting a tattoo, etc.

The signal does not affect the agents productivity. (This is a strong assumption made for simplicity and to emphasize the impact of signaling.)

The cost of the action for either type of worker is

$$c_H(y) = y > c_L(y) = \alpha y, \ \alpha \in (0, 1).$$

The Principal offers a wage to the agent after observing his action.

The Agency Problem: Signaling

Agent and principal face an incomplete information game Γ of five stages:

1. Nature determines the type of the agent matched to the Principal, either H or L, with probabilities q and $1 - q$.
2. The agent chooses his costly action, y.
3. The Principal observes y and makes a wage offer, w.
4. The Agent accepts or reject the wage offer.
5. Payoffs are realized.

The Agency Problem: Signaling

In the game Γ :

- A strategy for an agent of type $i \in \{H, L\}$ is $y_i \in [0, \bar{y}]$.
- A strategy for the principal is a mapping $w : [0, \bar{y}] \to \mathbb{R}_+$.
- A system of beliefs for the principal is a mapping $\mu : [0, \bar{y}] \to [0, 1]$, where $\mu(y) = \Pr(\text{Agent type is } H \mid y)$.

In a Perfect Bayesian equilibrium (PBE) of Γ:

- ▷ The Principal wage offers maximize her expected utility, and her beliefs are consistent with agents behavior.
- ▷ Each type of agent chooses optimally his signal given the Principal's wage offers, and responds optimally to the principal's wage offer.

The Agency Problem: Signaling

A separating (*signaling*) PBE of Γ :

$$\begin{aligned}
y_L &= 0,\ y_H = y^* \in (0, \bar{y}], \\
w^*(y) &= \begin{cases} x_L & y < y^* \\ x_H & y \geq y^* \end{cases}, \\
\mu^*(y) &= \begin{cases} 0 & y < y^* \\ 1 & y \geq y^* \end{cases}, \\
d_i(y, w) &= \begin{cases} \text{reject} & \text{if } w < x_i \\ \text{accept} & \text{if } w \geq x_i \end{cases}, \quad i \in \{L, H\}
\end{aligned}$$

The Agency Problem: Signaling

These strategies and belief system form an PBE provided the following inequalities hold:

(PC_H) $x_H - \alpha y^* \geq u_H$
(PC_L) $x_L \geq u_L$ (which holds by assumption)
(IC_H) $x_H - \alpha y^* \geq x_L$ (implied by PC_H since $u_H > x_L$ by assumption)
(IC_L) $x_L \geq x_H - y^*$.

By IC_L, y^* must satisfy

$$y^* \geq x_H - x_L$$

and by PC_H, y^* must satisfy

$$y^* \leq \frac{x_H - u_H}{\alpha}.$$

Hence

$$y^* \in \left[x_H - x_L, \frac{x_H - u_H}{\alpha}\right].$$

Since the agent action does not affect revenue, the most efficient signaling PBE is

$$y^* = x_H - x_L.$$

The Agency Problem: Signaling

In this signaling PBE agents surplus are $U_L^S = x_L - u_L$ and

$$\begin{aligned} U_H^S &= x_H - \alpha\left(x_H - x_L\right) - u_H \\ &= \alpha x_L + \left(1-\alpha\right) x_H - u_H \\ &= \bar{x}(\alpha) - u_H, \end{aligned}$$

Of course, this PBE exists only if $U_H^* \geq 0$, i.e., $\bar{x}(\alpha) \geq u_H$, or equivalently

$$\alpha \leq \frac{x_H - u_H}{x_H - x_L}.$$

That is, for a signaling PBE to exist the cost advantage of H agents must be significant.

There are also no-signaling PBE, i.e., such that

$$y_H = y_L = 0, \mu(\cdot) = q.$$

Specifically, there is a no-signaling PBE in which the principal offers the wage

$$\underline{w}(\cdot) = x_L.$$

And when $\bar{x}(q) > u_H$ there is also a no-signaling PBE in which the principal offers the wage

$$\bar{w}(\cdot) = \bar{x}(q).$$

The Agency Problem: Signaling

In these no-signaling PBEs agents surpluses are

$$\underline{U}_H^{NS} = 0, \underline{U}_L^{NS} = x_L - u_L = U_L^S$$

and

$$\bar{U}_i^{NS} = \bar{x}(q) - u_i \text{ for } i \in \{H, L\},$$

respectively.

The welfare comparison of alternative PBE yields interesting conclusions.

The Agency Problem: Signaling