Time and Uncertainty in Competitive Economies

Diego Moreno

Masters in Economics - Micro II
Universidad Carlos III de Madrid

Consider a pure exchange economy that operates over 2 consecutive dates, and in which the state of nature at date 2 is uncertain. As usual, we denote by l the number of (physical) goods, and by $S > 1$ the number of possible states of nature at date 2, and we let $\mathbb{R}^{l}_{+} \times \mathbb{R}^{lS}_{+}$ be the *commodity space.*

The characteristics of the consumers in the economy are described by the profile

$$[(u_1, \bar{x}_1, \bar{y}_1), ..., (u_n, \bar{x}_n, \bar{y}_n)],$$

where

$$u_i : \mathbb{R}^{l}_{+} \times \mathbb{R}^{lS}_{+} \to \mathbb{R}, \text{ and } (\bar{x}_i, \bar{y}_i) \in \mathbb{R}^{l}_{+} \times \mathbb{R}^{lS}_{+}$$

for each $i \in \{1, ..., n\}$.

An Arrow-Debreu Economy

In the Arrow-Debreu setting there is a complete set of markets where agents trade contingent contracts.

Hence, if the prices are $p = (p_x, p_y) \in \mathbb{R}^l_+ \times \mathbb{R}^{lS}_+$, the budget set of a consumer whose endowment is $(\bar{x}, \bar{y}) \in \mathbb{R}^l_+ \times \mathbb{R}^{lS}_+$ is

$$B(p, \bar{x}, \bar{y}) = \left\{ (x, y) \in \mathbb{R}^l_+ \times \mathbb{R}^{lS}_+ \mid p_x(x - \bar{x}) + \sum_{s=1}^{S} p_{ys}(y_s - \bar{y}_s) \leq 0 \right\}.$$

Note. In the inequality, we are multiplying vector prices and consumption bundles (i.e., the operation is an inner product). To avoid confusion, you may assume that there is one physical good ($l = 1$), consumption. With this assumption exchange is about allocating consumption across time (e.g., smoothing consumption) and states of nature (e.g., hedging risk).

An Arrow-Debreu CE is a collection

$$(p^*, x^*, y^*) \in \mathbb{R}_+^{l(1+S)} \times \mathbb{R}_+^l \times \mathbb{R}_+^{lS}$$

satisfying:

1. $\forall i$: $(x_i^*, y_i^*) \in \arg\max_{(x,y) \in B(p^*, \bar{x}_i, \bar{y}_i)} u_i(x, y)$,

2x. $\forall k$: $\sum_{i=1}^n (x_{ik}^* - \bar{x}_{ik}) \leq 0$ (with equality if $p_{xk}^* > 0$).

2y. $\forall (k, s)$: $\sum_{i=1}^n (y_{isk}^* - \bar{y}_{isk}) \leq 0$ (with equality if $p_{ysk}^* > 0$).

Arrow-Debreu CE allocations are PO (FWT)

Real Economies

In real economies there are spot markets and some contingent markets, although the set of markets is hardly complete, i.e., it is not possible to arrange contracts for the exchange of goods in some states of nature.

However, there are many real and financial assets that pay *returns* that depend on the state of nature.

Are the CE of these economies Pareto optimal?

A Radner Economy

Assume that there are spot markets at dates 1 and 2, and markets for *m securities* (real or financial assets) that operate at date 1.

Securities pay returns in units of good 1 (the numeraire) depending on the state of nature, as described by a matrix $R_{S\times m}$, whose columns specify the returns of each security.

(If there are the are contingent markets for some goods, they can be treated as securities, calculating the returns accordingly.)

In this economy, if $(\hat{p}, q) \in \mathbb{R}_+^{l(1+S)} \times \mathbb{R}_+^m$ is the vector of spot and security prices, then the budget set of a consumer whose endowment is $(\bar{x}, \bar{y}) \in \mathbb{R}_+^l \times \mathbb{R}_+^{lS}$ is

$$\begin{aligned}\hat{B}(\hat{p}, q, \bar{x}, \bar{y}) = \quad & \{(x, y, z) \in \mathbb{R}_+^l \times \mathbb{R}_+^{lS} \times \mathbb{R}^m \mid \\ & \hat{p}_x(x - \bar{x}) + \textstyle\sum_{j=1}^m \hat{q}_j z_j \leq 0, \\ & \forall s : \hat{p}_{ys}(y_s - \bar{y}_{is}) \leq \hat{p}_{ys1} \textstyle\sum_{j=1}^m R_{js} z_j\}.\end{aligned}$$

A Radner Economy

A Radner CE is a collection

$$(\hat{p}^*, q^*, x^*, y^*, z^*) \in \mathbb{R}_+^{l(1+S)} \times \mathbb{R}_+^m \times \mathbb{R}_+^l \times \mathbb{R}_+^{lS} \times \mathbb{R}^m$$

satisfying:

1. $\forall i$: $(x_i^*, y_i^*, z_i^*) \in \arg\max_{(x,y,z) \in \hat{B}(\hat{p}^*, q^*, \bar{x}_i, \bar{y}_i)} u_i(x, y)$,

2x. $\forall k$: $\sum_{i=1}^n (x_{ik}^* - \bar{x}_{ik}) \leq 0$ (with equality if $\hat{p}_{xk}^* > 0$).

2y. $\forall (k, s)$: $\sum_{i=1}^n (y_{isk}^* - \bar{y}_{isk}) \leq 0$ (with equality if $\hat{p}_{ysk}^* > 0$).

2z. $\forall j$: $\sum_{i=1}^n z_{ij}^* = 0$.

When are Radner CE allocations Pareto optimal?

Proposition. Assume $rank(R) = S$. If $(\hat{p}^*, q^*, x^*, y^*, z^*)$ is a Radner CE, then (x^*, y^*) is an Arrow-Debreu CE allocation, and therefore it is a Pareto optimal allocation.

Proof. For simplicity, assume that

$$R = I_{S \times S},$$

that is, there are S securities, and each security $s \in \{1, ..., S\}$ pays 1 unit of the numeraire in state s, and nothing in the other states. We show that (p^*, x^*, y^*), where $p^*_x = \hat{p}^*_x$ and

$$p^*_{ys} = q^*_s \frac{\hat{p}^*_{ys}}{\hat{p}^*_{ys1}},$$

is an Arrow-Debreu CE.

To this purpuse, we show that
(1) $(x_i^*, y_i^*) \in B(p^*, \bar{x}_i, \bar{y}_i)$, and
(2) $B(p^*, \bar{x}_i, \bar{y}_i) \subset \hat{B}_x(\hat{p}^*, q^*, \bar{x}_i, \bar{y}_i)$, where

$$\begin{aligned} \hat{B}_x(\hat{p}^*, q^*, \bar{x}_i, \bar{y}_i) = \quad & \{(x, y) \in \mathbb{R}_+^l \times \mathbb{R}_+^{lS} \mid \\ & (x, y, z) \in \hat{B}(\hat{p}^*, q^*, \bar{x}_i, \bar{y}_i) \text{ for some } z \in \mathbb{R}^S\}. \end{aligned}$$

Hence (x_i^*, y_i^*) maximizes u_i on $B(p^*, \bar{x}_i, \bar{y}_i)$. Also, since spot markets clear in the Radner equilibrium, and consumers' demands in the contingent markets equal those that clear spot markets, all contingent markets clear as well.

Therefore (p^*, x^*, y^*) is an Arrow-Debreu CE.

Uncertainty

To see that $(x_i^*, y_i^*) \in B(p^*, \bar{x}_i, \bar{y}_i)$, simply note that

$$\begin{aligned} p_x^*(x_i^* - \bar{x}_i) + \sum_{s=1}^{S} p_{ys}^* \left(y_{is}^* - \bar{y}_{is}\right) &= \hat{p}_x^*(x_i^* - \bar{x}_i) + \sum_{s=1}^{S} q_s^* \frac{\hat{p}_{ys}^*}{\hat{p}_{ys1}^*} \left(y_{is}^* - \bar{y}\right) \\ &\leq \hat{p}_x^*(x_i^* - \bar{x}_i) + \sum_{s=1}^{S} q_s^* \sum_{j=1}^{m} R_{js} z_{ij} \\ &= \hat{p}_x^*(x_i^* - \bar{x}_i) + \sum_{s=1}^{S} q_s^* z_{is} \\ &\leq 0. \end{aligned}$$

Uncertainty

Let $(x_i, y_i) \in B(p^*, \bar{x}_i, \bar{y}_i)$. We show that $(x_i, y_i, z_i) \in \hat{B}(\hat{p}^*, q^*, \bar{x}_i, \bar{y}_i)$, where

$$z_{is} = \frac{\hat{p}^*_{ys}}{\hat{p}^*_{ys1}} \left(y_{is} - \bar{y}_{is}\right).$$

for $s \in \{1, ..., S\}$. Thus,

$$\hat{p}^*_{ys}(y_{is} - \bar{y}_{is}) = \hat{p}^*_{ys1} z_{is} = \hat{p}^*_{ys1} \sum_{j=1}^{m} R_{js} z_{ij}.$$

Moreover,

$$\begin{aligned}
\hat{p}_x^*(x_i - \bar{x}_i) + \sum_{s=1}^{S} q_s^* z_{is} &= p_x^*(x_i - \bar{x}_i) + \sum_{s=1}^{S} q_s^* \frac{\hat{p}_{ys}^*}{\hat{p}_{ys1}^*} (y_{is} - \bar{y}_{is}) \\
&= p_x^*(x_i - \bar{x}_i) + \sum_{s=1}^{S} p_{ys}^* (y_{is} - \bar{y}_{is}) \\
&\leq 0.
\end{aligned}$$

This result generalizes to any setting in which the set of securities is rich enough to expand the opportunities for transferring income across dates and states in any possible way, i.e., when

$$rank(R) = S.$$

It is interesting to observe that securities may provide a more "economical" market structure than that required to have a complete set of contingent markets:

S	I	$S + 2I$	$I(1 + S)$
2	2	6	6
2	100	202	300
5	100	205	600
100	100	300	10, 100

Note: $T = 2$.

If $rank(R) < S$, then a CE of the Radner economy need not be Pareto optimal. In this case some interesting questions arise:

- Should Pareto optimality be defined relative to the market structure?
- Does creating more markets improve the efficiency of CE?