Exercise Set 4

1. There are two persons (A and B), two goods (quantities are x and y), and no production is possible. An allocation is a list $(x_A, y_A; x_B, y_B)$ specifying what each person receives of each good. The two persons' preferences over allocations are described by the utility functions $u^A(x_A, y_A; x_B, y_B) = 2x_A + y_A + \alpha \log x_B$, and $u^B(x_A, y_A; x_B, y_B) = x_B + y_B$. The two goods are available in the positive amounts w_x and w_y and α satisfies $0 < \alpha < w_x$. Note that Person A cares directly about how much Person B receives of the x-good. Determine all of the Pareto optimal allocations in which each person receives a positive amount of each good.

2. The de Beers Brewery uses water from the Pristine River in its brewing operations. Recently, the United Chemical Company (also called Chemco) has opened a factory upstream from de Beers. Chemco's manufacturing operations pollute the river water: let x denote the number of gallons of pollutant that Chemco dumps into the river each day. De Beers's profits are reduced by x^2 dollars per day, because that's how much it costs de Beers to clean the pollutants from the water it uses. Chemco's profit-maximizing level of operation involves daily dumping of 30 gallons of pollutant into the river. Altering its operations to dump less pollutant reduces Chemco's profit: specifically, Chemco's daily profit is reduced by the amount $\frac{1}{2}(30 - x)^2$ if it dumps x gallons of pollutant per day. There are no laws restricting the amount that Chemco may pollute the water, and no laws requiring that Chemco compensate de Beers for the costs imposed by Chemco's pollution.

(a) Coase's argument holds that the two firms will reach a bargain in which a Pareto efficient level of pollution will be dumped. Determine the efficient level of pollution. If efficiency requires that $x < 30$, then determine the range of bargains the two firms could be expected to reach– i.e., the maximum and minimum dollar amounts that de Beers could be expected to pay to Chemco in return for Chemco's agreeing to dump only x (less than 30) gallons per day.

(b) Now suppose a law is passed that requires anyone who pollutes the Pristine River to fully compensate any downstream firm for damages caused by the polluter's actions. How does this change the Pareto efficient level of pollution? How does it change the pollution level that Chemco and de Beers will agree to? How does it change the payments that one of the firms will make to the other?

(c) Now suppose that de Beers is not the only firm harmed by Chemco's pollution: There are two more brewing firms whose profits are reduced by the pollution in the same amount as de Beers. How would this affect your answers in (a) and (b)? (You will not be able to give a precise quantitative answer here, because you do not know exactly how much each firm is damaged by the pollution. But describe qualitatively how the answers to (a) and (b) will change.)

3. The tiny country of DeSoto has n households, each of which owns a car. The residents of DeSoto have only two interests in life, driving their cars and consuming the economy's only tangible commodity, simoleans. Each household has a utility function of the form $u^i(x^i, y^i) = y^i + v^i(x^i) - \alpha^i H$, where y^i denotes consumption of simoleans, x^i denotes miles driven, and H denotes the level of hydrocarbons in the air. Cars use simoleans for fuel: every mile that a car is driven uses up c simoleans, but the burning of simoleans also puts b units of hydrocarbons into the air for every mile that a car is driven. In other words, $H = b(x^1 + ... + x^n)$. Use A to denote the sum of all households' α^i parameters and x to denote the total of all miles driven by all households, and assume that each function v^i is strictly concave and increasing. Consider only those allocations in which each x^i and each y^i is positive. Each household has a positive endowment of simoleans.

(a) Give the n marginal conditions that characterize' the Pareto optimal allocations, and interpret them in words.

(b) Give the n marginal conditions that characterize the competitive equilibrium, and interpret them in words.

(c) Determine whether, in the equilibrium, all families necessarily drive "too much," all families necessarily drive "too little," or whether the miles driven might be either too large or too small, depending upon the data of the problem.

4. A certain restaurant is known for refusing to give separate checks to customers: After a group has ordered and eaten together at this restaurant, the group is presented with a single check for the entire amount that the group has eaten. It has been suggested that the restaurant does this because those who dine in groups will, with a single check, be more likely to simply divide the charge equally, each person paying the same amount irrespective of who ordered the most; and that diners, moreover, knowing that they will ultimately divide the charge equally, will order more than they would have ordered had each paid only for his own order. Analyze this situation using the following model. There are n diners in a group. Each has a utility function of the form $u(x^i, y^i) = y^i - \alpha^i \log x^i$, where x^i represents the amount of food (in pounds) ordered and eaten, and each y^i, represents the amount of money that i has after leaving the restaurant. The restaurant charges p dollars for each pound of food, and the restaurant's profit is an increasing function of the amount of food it sells at the price p. Each diner knows when he orders his food that the group will divide the check equally when it is time to pay. Compare the outcome under this check-splitting

arrangement with the outcome when each diner pays for his own order. Compare, in particular, the restaurant's profit in each case and the diners' welfare in each case. Is there an alternative arrangement that will make the diners better off than in either of these arrangements; what about the restaurant?

5. Ms. Alpha and Mr. Beta have just terminated their marriage. They have agreed that Mr. Beta will raise their only child, little Joey Alpha-Beta. The two parents hold no animosity toward one another, and each is concerned about little Joey's welfare. Their preferences are described by the utility functions $u^A(x, y^A) = x^\alpha y^A$ and $u^B(x, y^B) = x^\beta y^B$, where y^A and y^B denote the number of dollars "consumed" directly by the respective parents in a year, and x denotes the number of dollars per year consumed by Joey. Joey's consumption is simply the sum of the support contributions by his mother and his father, $s^A + s^B$. These contributions will be voluntary: Neither parent has sought a legal judgment against the other. Assume throughout that $\alpha = 1/4$ and $\beta = 1/3$.

(a) Suppose that Joey's mother is unable to contribute anything toward Joey's support, so that Mr. Beta must provide, out of his \$40,000 annual income, for both his own consumption y^B and Joey's consumption x. Express Mr. Beta's budget constraint both analytically and diagrammatically. Determine Mr. Beta's marginal rate of substitution between x and y^B at the choice he will make, and draw a diagram representing his choice problem. What levels of x and y^B will Mr. Beta choose?

(b) Actually, Ms. Alpha is going to contribute to Joey's support, but she is going to observe how much Mr. Beta contributes and then choose her contribution. Suppose that Mr. Beta does the same–i.e., each parent takes the other's contribution as given. If Ms. Alpha's annual Income is \$48,000 and Mr. Beta's is \$40,000, what will be their equilibrium contributions to Joey's support?

(c) Find an allocation of the parents' incomes that will make them both happier than the one in (b).

(d) Determine the two equations (*viz.*, the marginal condition and the "on-the-constraint" condition) that characterize the Pareto optimal allocations.

(e) Indicate some of the difficulties that a neutral third party (e.g., a judge) might encounter in attempting to implement some method for arriving at a Pareto optimal allocation of the parents' incomes.

6. 100 men have access to a common grazing area. Each man can choose to own either no cows, one cow, or two cows to provide milk for his family. The more cows the grazing land is required to support, the lower is each cowls yield of milk; specifically, a man who owns x_i cows obtains

$$Q_i = (250 - x)x_i$$

quarts of milk per year, where $x = \sum_{j=1}^{100} x_j$ is the total number of cows in the grazing land supports. Each man wants to obtain as much milk as he can, but no man has the resources to own more than two cows.

(a) How many cows do you predict each man will own? (Justify your answer.)

(b) Assume that the men can make transfers of milk among themselves. Is your prediction in (a) Pareto efficient for the 100 men? If so, verify it. If not, find a pattern of cow ownership and transfer payments that yields a Pareto optimal allocation of milk to the men that makes everyone strictly better off than in (a).

(c) Now suppose that there are only two men whose cows share a common grazing area, and that each cowls daily yield of milk, in quarts, depends on how many cows in total are grazing according to the following table

Total cows grazing:	1	2	3	4
Each cowls daily yield:	8	5	3	2

Which allocations of milk are Pareto efficient and individually rational (i.e., such that each man is at least as well off as he would be by "unilateral" action)? What are all the patterns of cow ownership and transfer payments that will support these allocations? Determine all the core allocations of milk to the two men.

(d) For the situation described in part (c), answer all the questions posed in part (a).