Existence of Equilibrium

First Approach, á la Arrow & Hahn

Let ℓ be a positive integer, and let $\mathcal{P}$ be a subset of $\mathbb{R}^\ell_+$. An *excess demand function* is a function $z\colon \mathcal{P}\to\mathbb{R}^\ell$.

Interpretation: There are ℓ goods, and each component p_k or $z_k(\cdot)$ represents a price or quantity of one of the goods. Each $p\in\mathcal{P}$ is a price list. For each $p\in\mathcal{P}$, $z(p)$ is a list of the excess (i.e., net) demands for each of the ℓ goods at the price list p.

We think of the demand function z as the sum of individual behavioral functions (i.e., demand and supply functions). We expect to ultimately show that a function $z(\cdot)$ defined by summing individual behavior will fit the approach we're developing. Thus, for the analysis we're doing now, we want to place restrictions on (i.e., make assumptions about) the function z that we will ultimately be able to verify by combining (a) theorems about individual behavior and (b) the summation of individual (consumer and firm) behavioral functions. This raises the question "what properties of individual behavior are preserved (or "inherited") under aggregation (i.e., summation)?" For example, is homogeneity in prices inherited? Budget balance? Continuity? Slutsky properites?

Definition: An equilibrium for z is a price list $p^*\in\mathcal{P}$ for which

(1) $z(p^*) \le \mathbf{0}$ and (2) $z_k(p^*) = 0$ if $p^*_k > 0$ $(k = 1,\ldots.\ell)$.

Motivation: We believe that such a p* will provide no forces to move the economy to some other price list p, and conversely, that other price lists p *will* lead to a change in p (for example, the prices of goods in excess demand rising and the prices of the goods in excess supply falling).

Notation: Let S denote the set $\{ p\in\mathbb{R}^\ell_+ \mid \sum_1^\ell p_k = 1 \}$, the unit simplex in $\mathbb{R}^\ell$.

Assumptions:

(A1) $S \subseteq \mathcal{P}$.

(A2) $\forall p\in\mathcal{P}\colon p\cdot z(p) = 0$. (Walras's Law)

(A3) z is continuous.

(Note that homogeneity of demand is not assumed; it is not used in the proof of existence.)

Remark: If $p^*\in\mathbb{R}^\ell_+$ and $z(p^*) \le \mathbf{0}$, and if z satisfies Walras's Law, then p^* is an equilibrium for z. (Before looking at the proof below, try proving it yourself. It's a relatively simple, but good, exercise.)

Proof: Since $p^* \ge \mathbf{0}$ and $z(p^*) \le \mathbf{0}$, each term of $p^*\cdot z(p^*)$ is nonpositive. Since the sum of the terms -- i.e., $p^*\cdot z(p^*)$ -- is zero, each term $p^*_k z_k(p^*)$ must be zero. Thus, for each k, either $z_k(p^*) = 0$ or else $p^*_k = 0$ (and in the latter case $z_k(p^*) \le 0$); in other words, p^* is an equilbrium. ■

Theorem: If $z:\mathcal{P}\to\mathbb{R}^\ell$ satisfies A1-A3, then it has an equilibrium.

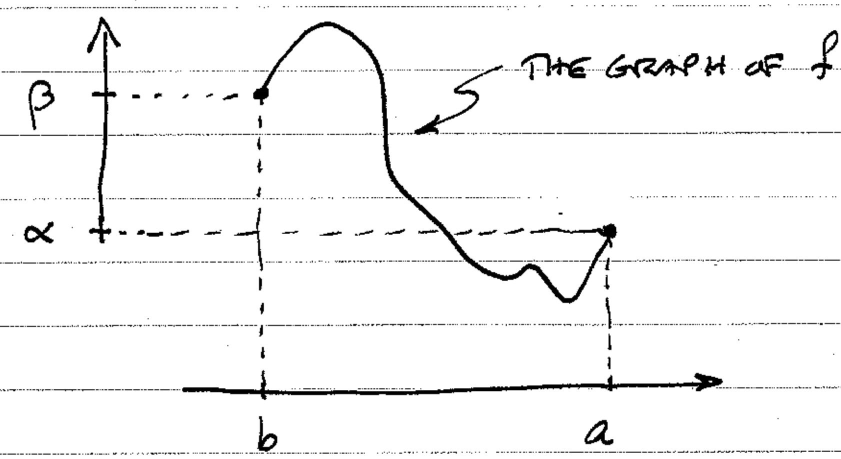
Proof (for ℓ=2): Let $\tilde{z}:[0,1]\to\mathbb{R}^2$ be defined by $\tilde{z}(p_1):= z(p_1,1-p_1)$; in other words, $\tilde{z}_1(p_1) = z_1(p_1,1-p_1)$ and $\tilde{z}_2(p_1) = z_2(p_1,1-p_1)$. Assumption A1 ensures that $\tilde{z}$ is well defined. Moreover, if there is a price $p^*_1\in [0,1]$ at which $\tilde{z}(p^*_1) \le \mathbf{0}$ (i.e., $\tilde{z}_k(p^*_1) \le 0$ for both k=1 and k=2), then the price list $(p^*_1, 1-p^*_1)$ is an equilibrium for z, because then $z_1(p^*_1, 1-p^*_1) \le 0$ and $z_2(p^*_1, 1-p^*_1) \le 0$. We therefore say that a price p^*_1 is an equilibrium if $\tilde{z}(p^*_1) \le \mathbf{0}$.

We first consider the two extreme points of [0,1], namely p_1=0 and p_1=1, and we show that if neither of them is an equilibrium, then we must have $\tilde{z}_1(0)>0$ and $\tilde{z}_2(1)>0$. At p_1=0, Walras's Law (A2) guarantees that $0\tilde{z}_1(0) + 1\tilde{z}_2(0) = 0$, and thus that $\tilde{z}_2(0) = 0$; therefore, if $\tilde{z}_1(0)\le0$, then p_1=0 is an equilibrium. The same argument establishes that that if $\tilde{z}_2(1)\le0$, then p_1=1 is an equilibrium. Consequently, if neither p_1=0 nor p_1=1 is an equilibrium, then both inequalities $\tilde{z}_1(0)>0$ and $\tilde{z}_2(1)>0$ must hold, and this will allow us show, by applying the Intermediate Value Theorem, that some $p_1\in(0,1)$ is an equilibrium. This argument goes as follows.

Since $\tilde{z}_2(1)>0$ and $\tilde{z}$ is continuous (this is A3), there is an interval $(a,1) \subseteq [0,1]$ on which $\tilde{z}_2(p_1)>0$. For each such p_1, Walras's Law guarantees (since $p_1>0$ and $1-p_1>0$) that $\tilde{z}_1(p_1)<0$. But we also have $\tilde{z}_1(0)>0$ (because p_1=0 is not an equilibrium); combining this with the fact that $\tilde{z}_1(p_1)<0$ for some $p_1>0$, which we have just established, and making use of the fact that $\tilde{z}_1$ is defined and continuous on $[0,p_1]$, we apply the Intermediate Value Theorem to obtain $\tilde{z}_1(p^*_1) = 0$ for some $p^*_1 \in (0,p_1)$. Again, Walras's Law guarantees that for such a p^*_1, we will also have $\tilde{z}_2(p^*_1) \le 0$ (in fact, since $p^*_2>0$, we have $\tilde{z}_2(p^*_1) = 0$). Therefore, p^*_1 is an equilibrium.

Summarizing, we have shown that if there is not an equilibrium in which either p_1=0 or p_2=0, then there is a p^*_1 which lies strictly between 0 and 1 at which $\tilde{z}_1(p^*_1) = 0$, which implies that $z(p^*_1, 1-p^*_1) = (0,0)$. ■

<u>THE INTERMEDIATE VALUE THEOREM:</u> LET f BE A CONTINUOUS REAL FUNCTION AND LET $f(a) = \alpha$ AND $f(b) = \beta$. FOR EVERY y BETWEEN α AND β [i.e., FOR EVERY "INTERMEDIATE VALUE" y] THERE IS AN x BETWEEN a AND b SUCH THAT $f(x) = y$.



<u>DEFN:</u> A SET X IS <u>DISCONNECTED</u> IF IT IS THE UNION OF TWO NONEMPTY DISJOINT SETS A AND B, EACH OF WHICH IS CLOSED IN X (i.e., $A = \tilde{A} \cap X$ AND $B = \tilde{B} \cap X$ FOR SOME PAIR OF CLOSED SETS $\tilde{A}$ AND $\tilde{B}$). A SET IS <u>CONNECTED</u> IF IT IS NOT DISCONNECTED.

<u>DEFN:</u> A SET A IS CLOSED IN X (OR "WITH RESPECT TO X") IF THERE IS A CLOSED SET $\tilde{A}$ FOR WHICH $A = \tilde{A} \cap X$.

Notes on the IVT Proof

The proof for ℓ=2 uses *all* the price lists p in the simplex S, and *only* those. In fact, what the proof requires is that z be defined for all the price lists p on some "downward sloping, connected curve," say C, from the vertical axis (p_1=0) to the horizontal axis (p_2=0 and $p_1=\bar{p}_1$, say), because then there is a simple continuous one-to-one identification of those price lists with the prices $p_1 \in [0,\bar{p}_1]$, which allows us to convert the problem to one involving two real functions z_1 and z_2 on the compact interval $[0,\bar{p}_1]$, and thus to apply the Intermediate Value Theorem (which applies only to *real* functions -- i.e., functions of a *single* real variable). Homogeneity of z in prices is not used. Note, however, that homogeneity does play an *informal* role: It assures us that if there *is* an equilibrium price list $p \in \mathbb{R}^{\ell}_{+}$, then there will also be an equilibrium $p' \in C$ (namely, λp for some $\lambda>0$); in other words, if there is an equilibrium, then limiting our search for it to the set C will not keep us from finding it.

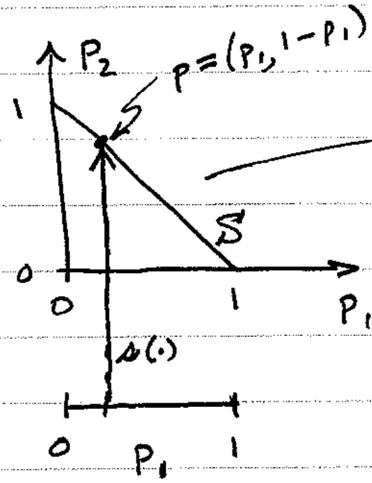
Can we adapt this proof for $\ell > 2$? For example, what about $\ell = 3$? We will have three prices, but by again considering only the price lists in the simplex S (which is now a two-dimensional subset of $\mathbb{R}^3$), we will only have to find a pair $(p_1,p_2) \in S' := \{ (p_1,p_2) \in \mathbb{R}^2_+ \mid p_1+p_2 \le 1 \}$ at which functions $\tilde{z}_1(\cdot)$, $\tilde{z}_2(\cdot)$, and $\tilde{z}_3(\cdot)$ defined on S' are all nonpositive. *But we will not be able to apply the Intermediate Value Theorem*, which holds only for real functions. The solution to this problem is to use a tool that is a generalization of the IVT to multivariate functions.

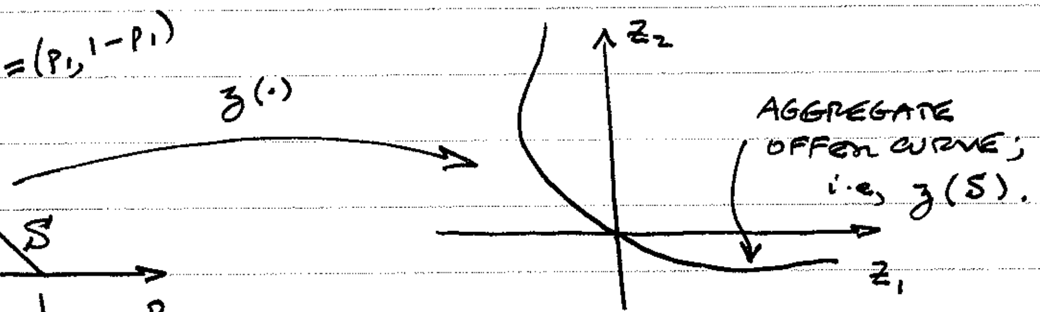
Brouwer's Theorem: If S is a nonempty compact convex set, and if $f: S \rightarrow S$ is a continuous function mapping S into itself, then f has a fixed point -- i.e., a point $x \in S$ for which $f(x) = x$.

Exercise: For each of the following conditions (assumptions) in Brouwer's Theorem, give a counterexample in which that condition (and *only* that one) is not satisfied, and in which the conclusion fails to hold: S is closed; S is bounded; S is convex; f is continuous.

The $\ell=2$ Proof Doesn't Adapt to $\ell>2$

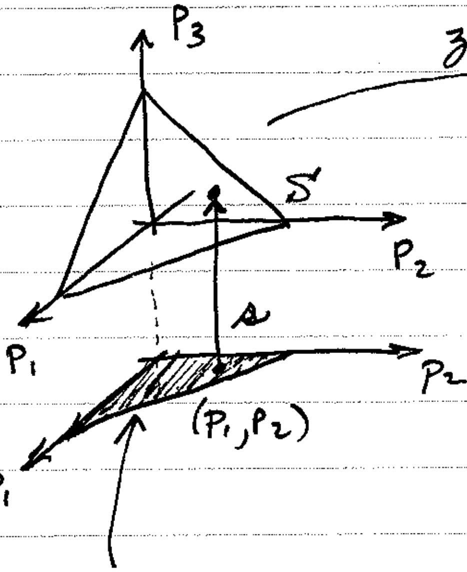
$\ell = 2$

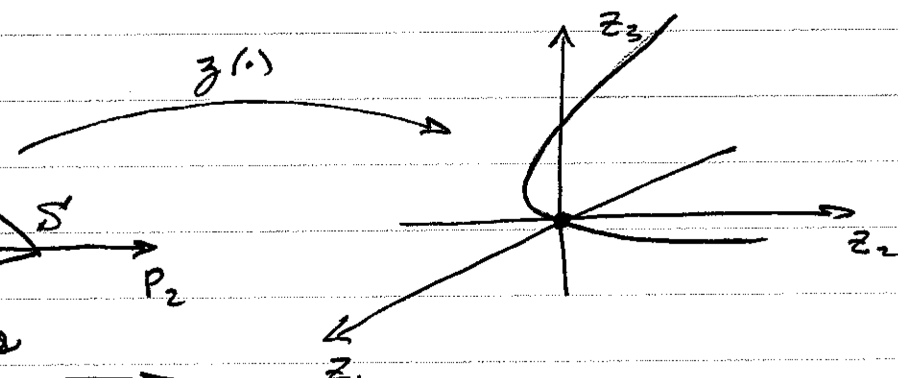




$$\tilde{z} := z \circ s \text{; i.e., } \tilde{z}(p_1) = z(s(p_1)) = z(p_1, p_2).$$

$\ell = 3$:





$$\tilde{z}(p_1, p_2) := z(p_1, p_2, 1-p_1-p_2) = z(s(p_1, p_2)).$$

$$\mathcal{S} = \{(p_1, p_2) \in \mathbb{R}^2_+ \mid p_1 + p_2 \leqq 1\}.$$

(★) ↑

(★) But IVT does not apply for $\mathcal{S} \subseteq \mathbb{R}^2$.

Equilibrium as a Fixed Point

If S is the set of states of a system, and if f is a function that describes the transition of the system from state to state -- i.e., if $s_{t+1} = f(s_t)$ -- then it is clear that we ought to simply define an equilibrium of the system as a fixed point of f. However, the equilibrium concept we have been using (a price list at which markets clear) was not derived from such a transition function, so we don't have a function f whose fixed points are the equilibria of our system. This is because we don't know precisely how prices will always move over time.

Nevertheless, our definition of equilibrium *was* motivated by an informal and *incomplete* notion of how prices move over time: *viz.*, if any of the goods are in excess demand [or supply], one or more of their prices will rise [or fall, if not already zero]. More simply, if there are any markets which are not cleared by the current prices, then those prices will change -- i.e., they are not equilibrium prices. And we assume that if, instead, the current prices *do* clear all the markets, then the prices will not change. In other words, while we don't know the precise transition function (or "laws of motion") of a market system, we make some weak assumptions about the properties of the transition function. These assumptions don't appear explicitly, but rather implicitly, in the definition of equilibrium that we use.

The usual approach to proving that an equilibrium exists is, in effect, to "go backwards" here. We have a definition of equilibrium that *would be* the fixed point(s) of *any* transition function f with the property that the price list changes if and only if there are some markets that don't clear at the current price list. So all we need to do is to *make up* such a transition function, and then show that it has a fixed point. This is precisely the way that most equilibrium existence proofs go. And this is true not just for Walrasian (competitive markets) equilibrium: It is usually the case that the definition of the equilibrium of a system formalizes the idea that the system is *stationary* under some incompletely specified (because incompletely understood) dynamics -- i.e., that the equilibrium states are the fixed points of *any* of a ceratain *class* of dynamic processes, or transition functions.

Theorem: If $z:\mathcal{P}\to\mathbb{R}^\ell$ satisfies A1-A3, then it has an equilibrium.

Proof: Let $f:S\to S$ be defined by

$$f(p) = \frac{1}{\sum_1^\ell [p_i + M_i(p)]} [\, p + M(p) \,] \quad , \quad \text{for every} \quad p \in S,$$

where $M_i(p) := \max \{ 0, K_i z_i(p) \}$, $i = 1,\ldots,\ell$, for some positive numbers $K_1,\ldots,K_\ell$. We will show that

(a1) f indeed maps S into S,

and (a2) f is continuous,

from which it follows (via Brouwer's Theorem) that

(a) f has a fixed point.

Then we will show that

(b) every fixed point of f is an equilibrium of z.

Proof of (a1): For each $p \in S$ and for each i, we have $M_i(p) \geq 0$ (by definition of $M_i(p)$), and therefore $p_i + M_i(p) \geq 0$. Furthermore, since $p_i > 0$ for some i, we have $p_i + M_i(p) > 0$ for that i, and therefore $\sum [\, p_i + M_i(p) \,] > 0$, which guarantees that f(p) is defined and in S.

Proof of (a2): Each function $M_i(\cdot)$ is clearly continuous, and therefore f is continuous.

Proof of (b): Suppose p is a price list for which $f(p) = p$ -- i.e.,

$$p + M(p) = \textstyle\sum_1^\ell [\, p_i + M_i(p) \,]\, p.$$

Then $pz(p) + M(p)z(p) = \sum_1^\ell [\, p_i + M_i(p) \,]\, p\, z(p)$, and application of Walras's Law (A2) yields $M(p)z(p) = 0$. Now consider in turn each term $M_i(p)z_i(p)$ in the sum $M(p)z(p)$: If $z_i(p) > 0$, then $M_i(p) > 0$, and therefore $M_i(p)z_i(p) > 0$; and if $z_i(p) \leq 0$, then $M_i(p) = 0$, and therefore $M_i(p)z_i(p) = 0$. Thus, each term is nonnegative; and the zero terms are precisely the ones for which $z_i(p) \leq 0$. Since the sum is zero, each term must in fact be zero, and therefore $z_i(p) \leq 0$ for *every* i -- in other words, p is an equilibrium. ■

Notes on the Fixed-Point Proof

As Arrow and Hahn indicate, since only a few of the properties of the function $M(\cdot)$ are actually used in the proof, *any* price-dynamics function $M(\cdot)$ that has those properties would work just as well. But it is also important to notice that no assumption is being made here about what the "transition" function f *really* is -- i.e., about how prices *really* move over time. We are merely *making up* a transition function that will have the properties (a) and (b), if we can, because if there *is* such a function then (a) it will have a fixed point, which (b) will be an equilibrium.

Note that Assumptions A1, A2, and A3 were all used: A1 in the proof of (a1) -- if S were replaced by any proper subset S′ of S, we would not be sure that $f(S') \subseteq S'$; A2 in the proof of (b); and A3 in the proof of (a2).

Some Problems with the Approach So Far

1. The function $z(\cdot)$ could not have come from an economy in which some agents have constant returns to scale or "flat spots" on their indifference surfaces, because then z would not be single-valued. The way we deal with this is to use *correspondences* (multi-valued functions) instead of single-valued functions.

2. The function $z(\cdot)$ could not have come from an economy in which some consumers have strictly increasing preferences, because then $z(p)$ would not be defined for price lists in which some of the prices are zero. We might try to deal with this by restricting the domain of z to some compact subset $\mathcal{P}'$ of $\mathcal{P}$ on which z *is* always defined. But this is not so simple: The set $\mathcal{P}'$ must be constructed in such a way that the equilibrium price list has not been omitted from it! (It should also be noted that when nonmonotone preferences are allowed, individual budget balance becomes an inequality, and therefore so does Walras's Law.)