March 29, 2019

MATHEMATICS FOR ECONOMICS II (2018-19)

ECONOMICS, LAW–ECONOMICS, INTERNATIONAL STUDIES–ECONOMICS

SHEET 5. SEQUENCES AND SERIES

(1) Study whether the following sequences converge, and if they converge, find the limit.

(a) $(-1)^n$ (b) $\frac{3^n+6\cdot 5^n}{5^n+e^n}$ (c) $\cos(0.7^n)$ (d) $\frac{n}{n+1}\cos(n\pi)$

(e) $n^{(-1)^n}$ (f) $\frac{n^2-n}{n-\sqrt{n}}$, $n \geq 2$ (g) $\frac{1+2+\cdots+n}{n^\alpha}$, $\alpha \in \mathbb{R}$ (h) $\cos^n(0.7)$

(i) $\left(1+\frac{2}{n}\right)^n$ (j) $\sqrt[n]{n^2}$ (use that $n^{1/n} \to 1$) (k) $\frac{\ln n^\alpha}{n}$, $\alpha \in \mathbb{R}$

(l) x_n, where $x_{n+1} = \sqrt{1+x_n}$, for $n \geq 1$, with $x_1 = 0$.

(m) x_n, donde $x_{n+1} = \sqrt{14+\sqrt{x_n}}$, para $n \geq 1$, con $x_1 = 0$.

(n) It is known that the sequence $x_n = \sqrt[n]{c}$ converges, where $c \geq 1$. Which is the limit?

(ñ) Calculate the limit of the sequence $x_n = \sqrt[n]{c}$, where $0 < c < 1$.

Answers: When the sequence converges, the value of the limit is given. (a) Not convergent; (b) 6; (c) 1; (d) not convergent; (e) not convergent; (f) not convergent; (g) 2 if $\alpha = 2$, 0 if $\alpha > 2$, not convergent if $\alpha < 2$; (h) 0; (i) e^2; (j) 1; (k) 0; (l) $\frac{1+\sqrt{5}}{2}$: (m) 4; (n) 1; (ñ) 1.

(2) Decide whether the following sequences are convergent and discuss the associated series.

(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots.$

(b) $\frac{3}{2}, \frac{6}{3}, \frac{9}{4}, \frac{12}{5}, \ldots.$

Answers: (a) The sequence converges to 1, thus the series diverges; (b) The sequence converges to 3, thus the series diverges.

(3) Study the convergence of the following positive series.

(a) $1 + 0.9 + 0.81 + 0.729 + 0.6561 + \ldots\ldots$

(b) $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots\ldots$

(c) $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \ldots\ldots$

(d) $\sum_{n=1}^{\infty} \frac{(n+1)^2}{3n^2+7}$ (e) $\sum_{n=1}^{\infty} \frac{1}{2^{\frac{n}{2}}}$ (f) $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$

(g) $\sum_{n=1}^{\infty} \frac{3}{n^2+7}$ (h) $\sum_{n=1}^{\infty} \frac{e^n}{n^n}$ (i) $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

(j) $\sum_{n=2}^{\infty} \frac{1}{n^2\ln n}$ (k) $\sum_{n=1}^{\infty} \frac{n^2}{n!}$ (l) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

(m) $\sum_{n=0}^{\infty} 3(\frac{3}{2})^n$ (n) $\sum_{n=0}^{\infty} \frac{2^n+1}{2^{n+1}}$ (ñ) $\sum_{n=0}^{\infty} \frac{n+10}{10n+1}$

(o) $\sum_{n=6}^{\infty} \frac{1}{n^2+1}$ (p) $\sum_{n=0}^{\infty} e^{-n}$ (q) $\sum_{n=2}^{\infty} (\frac{\ln n}{n})^n$

(r) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ (s) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ (t) $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$

(u) $\sum_{n=1}^{\infty} \frac{n!}{3^n}$ (v) $\sum_{n=1}^{\infty} \frac{n!}{2^{2n}}$ (w) $\sum_{n=1}^{\infty} (n^{\frac{1}{n}} - 1)^n$

(x) $\sum_{n=1}^{\infty} e^{-n^2}$

Answers: (a) The sum equals 10; (b) divergent; (c) convergent; (d) divergent; (e) the sum equals $\frac{1}{\sqrt{2}-1}$; (f) convergent; (g) convergent; (h) convergent; (i) divergent; (j) convergent; (k) convergent; (l) divergent; (m) divergent; (n) divergent; (ñ) divergent; (o) divergent; (p) convergent; (q) convergent; (r) divergent; (s) convergent; (t) convergent; (u) divergent; (v) divergent; (w) convergent; (x) convergent.

(4) Prove the following identities:

(a) $\sum_{n=1}^{\infty} [\frac{1}{2n-1} - \frac{1}{2n+1}] = 1$ (b) $\sum_{n=1}^{\infty} \frac{2}{3^{n-1}} = 3$ (c) $\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{3}{4}$

Answers: (a) It is a telescoping series; (b) geometric of ratio 1/3 and first term 2.

(5) Calculate the sum of the following series:

(a) $\sum_{n=1}^{\infty} \frac{1+3^n}{5^n}$ (b) $\sum_{n=1}^{\infty} (-\frac{1}{3})^{n+1}$ (c) $\sum_{n=2}^{\infty} \frac{2}{(n-1)(n+1)}$

Answers: (a) $\frac{7}{4}$; (b) $\frac{1}{16}$; (c) $\frac{3}{2}$.

(6) Study the conditional and absolute convergence of the following series.

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{3^n}$ (b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2+1}{2n^2+3}$ (c) $\sum_{n=2}^{\infty} (-1)^{n-1} \frac{\ln n}{n}$

(d) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ (e) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^3+1}$ (f) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{1+n^2}$

(g) $\sum_{n=1}^{\infty} \frac{sen(n\theta)}{n^2}$ (h) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}$

Answers: (a) Converges absolutely; (b) diverges; (c) converges conditionally; (d) converges conditionally; (e) Converges absolutely; (f) diverges; (g) Converges absolutely; (h) Converges absolutely.

(7) For each of the series below, give an approximated value of its sum by using the 6 first terms of the series, and provide an estimate of the error.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$ (b) $\sum_{n=1}^{\infty} \frac{4(-1)^n}{2n+1}$

Answers: (a) $S_6 = 0.6319$, $|S - S_6| < 0.0002$; (b) $S_6 = -0.7163$, $|S - S_6| < 0.2667$.

(8) Let $\{a_n\}$ be a sequence with $a_n > 0$ that converges to 0. What can be said about the convergence of the following series?

(a) $\sum_{n=1}^{\infty} (1 + a_n)$.

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2+a_n}$.

(c) $\sum_{n=0}^{\infty} e^{-n(1+a_n)}$.

Answers: (a) Diverges; (b) converges; (c) converges.