March 21, 2019

MATHEMATICS FOR ECONOMICS II (2018-19)

ECONOMICS, LAW–ECONOMICS, INTERNATIONAL STUDIES–ECONOMICS

SHEET 4. PRIMITIVES AND INTEGRALS 2

(1) Find:

$$a)\int_0^1 \frac{1}{\sqrt{x}}dx \quad b)\int_0^3 \frac{1}{x^3}dx \quad c)\int_1^\infty \frac{1}{x^2}dx$$
$$d)\int_1^\infty e^{-x}dx \quad e)\int_{-\infty}^\infty \frac{dx}{1+x^2} \quad f)\int_{-2}^4 \frac{dx}{x^2}$$

Answer: a) 2; b) ∞, c) 1; d) 1; e) π; f) ∞.

(2) Find $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(1+x)}$.

Answer: π.

(3) Compute $\displaystyle\int_1^\infty (1-x)e^{-x}\, dx$.

Answer: $-\frac{1}{e}$.

(4) Let $f:\mathbb{R}\to\mathbb{R}$ a function with a continuous derivative f' defined on $\mathbb{R}$. Given an interval $[a,b]$, consider the body of revolution S obtained by rotating the graph of f about the X-axis. The surface area and volume of S can be computed with the formulas,

$$\text{area}(S) = 2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx, \quad \text{volume}(S) = \pi\int_a^b (f(x))^2\,dx$$

Compute the surface area and the volume obtained by revolving the graph of the function $f(x) = \dfrac{1}{x}$ about the X-axis in the interval $[1,\infty)$.

Answer: The volume is π but the surface area is ∞.

(5) Let

$$I(\alpha) = \int_0^\infty e^{-t}\frac{\sin \alpha t}{t}\,dt$$

(a) Show that $I(0)=0$.
(b) Compute $I'(\alpha)$.
(c) Use the previous parts to compute $I(\alpha)$.

Answer: $I'(\alpha) = \frac{1}{1+\alpha^2}$, $I(\alpha) = \arctan\alpha$.

(6) Let $I(\alpha) = \displaystyle\int_0^\infty \frac{e^{-\alpha t}}{t}\,dt$, $\alpha > 0$.
(a) Compute $I'(\alpha)$.
(b) Compute $I(\alpha)$.

(c) Compute $J(\alpha,\beta) = \displaystyle\int_0^\infty \frac{e^{-\alpha t} - e^{-\beta t}}{t}\, dt$, $\alpha, \beta > 0$.

Answer: $I'(\alpha) = -\frac{1}{\alpha}$, $I(\alpha) = -\ln\alpha + C$, C constant, $J(\alpha,\beta) = I(\alpha) - I(\beta) = \ln\frac{\beta}{\alpha}$.

(7) Let $I(t) = \displaystyle\int_0^\infty e^{-tx}\, dx$, $t > 0$.

(a) Prove that $I(t) = \dfrac{1}{t}$.

(b) Prove that $\displaystyle\int_0^\infty x^n e^{-tx}\, dt = \frac{n!}{t^{n+1}}$.

(c) Prove that $n! = \displaystyle\int_0^\infty x^n e^{-x}\, dt$.

Answer:

(8) Let $I(\alpha) = \displaystyle\int_0^\infty e^{-\alpha t}\frac{\sin t}{t}\, dt$, $\alpha > 0$.

(a) Compute $\lim_{\alpha\to\infty} I(\alpha)$.

(b) Compute $I'(\alpha)$.

(c) Use the previous parts to compute $I(\alpha)$.

(d) Compute $\displaystyle\int_0^\infty \frac{\sin t}{t}\, dt$.

(e) Compute $\displaystyle\int_{-\infty}^\infty \frac{\sin t}{t}\, dt$.

Answer: (a) 0; (b) $I'(\alpha) = -\frac{1}{1+\alpha^2}$; (c) $I(\alpha) = \frac{\pi}{2} - \arctan\alpha$; (d) $\frac{\pi}{2}$; (e) π.