May 6, 2019

MATHEMATICS FOR ECONOMICS II (2018-19)

ECONOMICS, LAW–ECONOMICS, INTERNATIONAL STUDIES–ECONOMICS

SHEET 3. PRIMITIVES AND INTEGRALS

(1) *Find the following integrals:*

$$
\begin{array}{lll}
a) \displaystyle\int \frac{x^2+x+1}{x\sqrt{x}}dx & b) \displaystyle\int xe^{-2x}dx & c) \displaystyle\int \sin^{14} x\ \cos x\ dx \\
d) \displaystyle\int (x+1)(2-x)^{1/3}dx & e) \displaystyle\int \frac{x^4}{1+x^5}dx & f) \displaystyle\int e^{2x}\ \sin x\ dx \\
g) \displaystyle\int \frac{1}{3+x^2}dx & h) \displaystyle\int x\cos x\ dx & i) \displaystyle\int \left(1+\frac{1}{x}\right)^3 \frac{1}{x^2}\ dx \\
j) \displaystyle\int x\sin ax^2 dx & k) \displaystyle\int \frac{x}{\sqrt{16-x^2}}\ dx & l) \displaystyle\int \frac{1}{\frac{x^2}{2}-2x+4}\ dx \\
m) \displaystyle\int \frac{40x}{(x-1)^{40}}\ dx & n) \displaystyle\int \frac{x^2+1}{x^3-4x^2+4x}\ dx & o) \displaystyle\int \frac{2x+1}{x^3+6x}\ dx \\
p) \displaystyle\int \frac{x^4}{x^4-1}\ dx & q) \displaystyle\int \frac{4x+6}{(x^2+3x+7)^3}dx & r) \displaystyle\int \frac{2x-6}{(x-2)^2}dx
\end{array}
$$

Solution: a) $\frac{2}{3}x^{3/2}+2\sqrt{x}-\frac{2}{\sqrt{x}}+C$; Write $I=\int(x^2+x+1)x^{-3/2}dx=\int x^{1/2}+x^{-1/2}+x^{-3/2}dx$ and integrate term by term.

b) $-\frac{e^{-2x}(2x+1)}{4}+C$; By parts, $u=x$, $dv=e^{-2x}dx$, so that $du=dx$ and $v=-\frac{1}{2}e^{-2x}$. Then use $\int udv=uv-\int vdu$.

c) $\frac{\sin^{15} x}{15}+C$; Change of variable, $t=\sin x$, so that $dt=\cos x dx$. Hence $I=\int t^{14}dt$.

d) $-\frac{3(2-x)^{4/3}(4x+13)}{28}+C$; Change of variable, $t=2-x$, so that $dt=-dx$ and $x+1=t+3$. Hence $I=-\int(t+3)t^{1/3}dt=-\int t^{4/3}+3t^{1/3}dt$ and integrate term by term.

e) $\frac{\ln|1+x^5|}{5}+C$; Change of variable, $t=x^5$, so that $dt=5x^4dx$ and thus $I=\int\frac{1/5}{1+t}dt$ and integrate.

f) $\frac{e^{2x}}{5}(2\sin x-\cos x)+C$; By parts, $u=e^{2x}$ and $dv=\sin x dx$, so that $du=2e^{2x}dx$ and $v=-\cos x$. We find

$$I=-2e^{2x}\cos x+2\int e^{2x}\cos x dx.$$

For the second integral take parts again ($u=e^{2x}$ and $dv=\sin x dx$, thus $du=2e^{2x}dx$, $v=\sin x$) to find

$$
\begin{aligned}
I &= -2e^{2x}\cos x+2\int e^{2x}\cos x dx = -2e^{2x}\cos x+2\left(e^{2x}\sin x-2\int e^{2x}\sin x dx\right) \\
&= -2e^{2x}\cos x+2e^{2x}\sin x-4I.
\end{aligned}
$$

Solving for I we get the result.

g) $\frac{\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}x}{3}\right)+C$; Divide both numerator and denominator by 3 to get $I=\frac{1}{3}\int\frac{1}{1+(x/\sqrt{3})^2}dx$. Then recall that the antiderivative is an arctangent.

h) $\cos x+x\sin x+C$; By parts, $u=x$, $dv=\cos x dx$.

i) $-\frac{(1+\frac{1}{x})^4}{4}+C$; Change of variable $t=x^{-1}$, so $dt=-x^{-2}dx$. Thus $I=-\int(1+t)^3dt=-(1+t)^4/4+C$ and so on.

j) $-\frac{\cos ax^2}{2a}+C$; Change of variable, $t=ax^2$, thus $dt=2axdx$ and $I=\frac{2a}{\int}\sin tdt$.

k) $-\frac{\sqrt{16-x^2}}{2}+C$; Change of variable $t=16-x^2$, $dt=-2x$. Thus $I=-2\int\frac{1}{\sqrt{t}}dt$.

l) $\arctan\left(\frac{x-2}{2}\right) + C$; (Hard). Write the denominator $x^/2 - 2x + 4 = \frac{1}{2}((x-2)^2 + 4) = 2\left(\left(\frac{x-2}{2}\right)2 + 1\right)$. Hence $I = \int \frac{1/2}{\left(\frac{x-2}{2}\right)2+1} dx = \int \frac{dt}{t^2+1}$.

m) $\int \frac{40x}{(x-1)^{40}}\,dx = \int \frac{40(x-1)}{(x-1)^{40}}\,dx - \int \frac{40}{(x-1)^{40}}\,dx = -\frac{40}{38}(x-1)^{-38} - \frac{1}{39}(x-1)^{-39} + C$

n) $\int \frac{x^2+1}{x^3-4x^2+4x}\,dx = \frac{1}{4}\int \frac{1}{x}\,dx + \frac{3}{4}\int \frac{1}{x-2}\,dx + \frac{5}{2}\int \frac{1}{(x-2)^2}\,dx = \frac{1}{4}\ln x + \frac{3}{4}\ln(x-2) - \frac{5}{2}(x-1)^{-1} + C$

0) $\int \frac{2x+1}{x^3+6x}\,dx = \int \left(\frac{2-\frac{x}{6}}{x^2+6} + \frac{1/6}{x}\right)\,dx = -\frac{1}{12}\int \frac{2x}{x^2+6}\,dx + 2\int \frac{1}{x^2+6}\,dx + \frac{1}{6}\ln x + C =$
$= -\frac{1}{12}\ln(x^2+6) + \frac{2\sqrt{6}}{6}\arctan\frac{x\sqrt{6}}{6} + \frac{1}{6}\ln x + C$

p) $\int \frac{x^4}{x^4-1}\,dx = \int \left(1 + \frac{1}{x^4-1}\right)\,dx = x + \int \frac{1}{x^4-1}\,dx = x + \int \left(\frac{1/4}{x-1} - \frac{1/4}{x+1} - \frac{1/2}{x^2+1}\right)\,dx = x + \frac{1}{4}\ln(x-1) - \frac{1}{4}\ln(x+1) - \frac{1}{2}\arctan x + C$

q) $\displaystyle\int \frac{4x+6}{(x^2+3x+7)^3}dx = 2\int (2x+3)(x^2+3x+7)^{-3}dx = 2(x^2+3x+7)^{-2}\ /(-2) + C$

p) $\displaystyle\int \frac{2x-6}{(x-2)^2}dx = \int \frac{2(x-2)-2}{(x-2)^2}dx = \int (\frac{2}{(x-2)} - \frac{2}{(x-2)^2})dx = 2\ln(x-2) + \frac{2}{x-2} + C$

(2) *Evaluate $F'(x)$ in the following cases:*

$$a)\ \int_1^x \left(t^2 - 2t + 5\right)\,dt \quad b)\ \int_x^0 t\cos t\,dt \quad c)\ x\left(\int_x^0 t\cos t\,dt\right)$$

Solution: a) $x^2 - 2x + 5$; b) $-x\cos x$; c) $\left(\int_x^0 t\cos t dt\right) - x^2\cos x$.

(3) *Consider the function $F(x) = \displaystyle\int_{-3}^x \frac{t^2-4}{3t^2+1}\,dt$.*
   (a) *Find the local maximum points of $F(x)$. Is any of these points a global maximum?*
   (b) *Find the local minimum points of $F(x)$.*
   Let now $G(x) = \displaystyle\int_{-1}^x \frac{t^2-4}{3t^2+1}\,dt$. Does $G(x)$ have a global minimum?

Solution: $F'(x) = \frac{x^2-4}{3x^2+1}$, thus $x = \pm 2$ are the only critical points. Note that F' is positive for $x < -2$, negative for $-2 < x < 2$ and positive again for $x > 2$, thus -2 is a local maximum and 2 is a local minimum. By the way, it is a good exercise to compute F'' and then conclude that $x = 0$ is an inflection point.

(4) *In each case, find the area of the figure bounded by the functions f and g.*

$$a)\ f(x) = x^2 - 4x + 3, \quad g(x) = -x^2 + 2x + 3$$
$$b)\ f(x) = (x-1)^3, \quad g(x) = x - 1$$
$$c)\ f(x) = x^4 - 2x^2 + 1, \quad g(x) = 1 - x^2$$

Solution: a) 9; b) 0.5; c) 4/15.

(5) *Draw the functions $y = 2e^{2x}$ and $y = 2e^{-2x}$ and find the area bounded by these two functions and the lines $x = -1$, $x = 1$.*

Solution: $2(e - e^{-1})^2 \approx 11.0488$.

(6) *Find the tangent line to the graph of $f(x)=\sqrt{x}$ at the point $x=4$ and calculate the area of the region enclosed between the graph of f and its tangent line, and the lines $x=0$ and $x=4$.*

Solution: $y=1+\frac{x}{4}$; $A=\frac{2}{3}$.

(7) *An asset X pays dividends $D(t)dt$ at instant of time t. The total present value of dividends in the interval $[0,T]$, $T>0$, is*

$$V(0)=\int_0^T e^{-rt}D(t)dt,$$

where $r>0$ is the continuous rate of interest of a riskless government bond in the same period. Find $V(0)$ in the following cases.

(a) $D(t)=1$.
(b) $D(t)=2$ *up to* $\frac{T}{2}$ *and* $D(t)=0$ *in* $(\frac{T}{2},T]$.
(c) $D(t)=e^{it}$, *where* $i>0$.
(d) $D(t)=\sin\frac{\pi t}{T}$ *(harder).*

Solution: a) $\frac{1}{r}(1-e^{-rT})$; b) $\frac{2}{r}(1-e^{-rT/2})$; c) $\frac{1}{r-i}(1-e^{(i-r)T})$, if $i\neq r$ and T otherwise; d) $(1+e^{-rT})\frac{\frac{T}{\pi}}{1+(\frac{rT}{\pi})^2}$.

(8) *Let $f:[0,2]\longrightarrow\mathbb{R}$ be continuous, increasing in $(0,1)$, decreasing in $(1,2)$ and, also, satisfying that: $f(0)=3$, $f(1)=5$ and $f(2)=4$. Between which values can we guarantee that $\int_0^2 f(x)\,dx$ is located?*

Solution: Since, $3\leq\int_0^1 f(x)\,dx\leq 5$, and $4\leq\int_1^2 f(x)\,dx\leq 5$, we have that $7\leq\int_0^2 f(x)\,dx\leq 10$.

(9) *Let $f:[1,3]\longrightarrow[2,4]$ be increasing, continuous and bijective such that $\int_1^3 f\,dx=5$. Calculate $\int_2^4 f^{-1}(x)\,dx$*

Solution: Since $\int_1^3 f\,dx+\int_2^4 f^{-1}(x)\,dx=10$, $\int_2^4 f^{-1}(x)\,dx=10-5=5$.

(10) *Certain company has determined that its marginal cost is $\frac{dC}{dx}=4(1+12x)^{-1/3}$. Find the cost function if $C=100$ when $x=13$.*

Solution: $\frac{1}{3}(1+12x)^{2/3}$.