February 20, 2020

# MATHEMATICS FOR ECONOMICS II (2018-19)

*ECONOMICS, LAW–ECONOMICS, INTERNATIONAL STUDIES–ECONOMICS*

## SHEET 2. DIAGONALIZATION AND QUADRATIC FORMS

(1) *Given the matrix*

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$$

*compute its eigenvalues, eigenvectors and diagonalize $A$.*

**Solution:** The characteristic polynomial is

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 4 \\ 3 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 10$$

The roots are

$$\lambda = \frac{3 \pm \sqrt{9+40}}{2} = -2, 5$$

There are two different eigenvalues. The matrix is diagonalizable.

The eigenspace $S(5)$ is the solution to the system of linear equations

$$\begin{pmatrix} -3 & 4 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

which is equivalent to the following one

$$3x - 4y = 0.$$

Therefore, $S(5) < (4,3) >$.

The eigenspace $S(-2)$ is the solution to the system of linear equations

$$\begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

which is equivalent to the following one

$$x + y = 0.$$

Therefore, $S(-2) =< (1,-1)$.

We conclude that $A = PDP^{-1}$ with

$$D = \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix} \qquad P = \begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix}$$

(2) *Given the following matrices*

$$A = \begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix} \qquad C = \begin{pmatrix} 4 & 5 & -2 \\ -2 & -2 & 1 \\ -1 & -1 & 1 \end{pmatrix}$$

(a) *Compute its eigenvalues, eigenvectors and the eigenspaces.*
(b) *Diagonalize them, whenever possible.*

**Solution:**

First, we find the eigenvalues of $A$. The characteristic polynomial is

$$|A - \lambda I| = \begin{pmatrix} 4-\lambda & 6 & 0 \\ -3 & -5-\lambda & 0 \\ -3 & -6 & 1-\lambda \end{pmatrix} = -(\lambda+2)(\lambda-1)^2$$

so the eigenvalues ar $\lambda_1 = -2$ with multiplicity $n_1 = 1$ and $\lambda_2 = 1$ with multiplicity $n_2 = 2$.

Now we compute the eigenspace $S(1)$. We solve the following system of linear equations

$$(A - I)\begin{pmatrix} x & y & z \end{pmatrix} = \begin{pmatrix} 3 & 6 & 0 \\ -3 & -6 & 0 \\ -3 & -6 & 0 \end{pmatrix}\begin{pmatrix} x & y & z \end{pmatrix} = 0$$

o sea,

$$\begin{aligned} 3x + 6y &= 0 \\ -3x - 6y &= 0 \end{aligned}$$

using $y$ as the parameter, the solution is $S(1) = \{-2y, y, z) : y, z \in \mathbb{R}\} =< (-2, 1, 0), (0, 0, 1) >$ so $\dim S(1) = 2$.

On the other hand, $S(-2)$ is the set of solutions to the system of linear equations

$$(A + 2I)\begin{pmatrix} x & y & z \end{pmatrix} = \begin{pmatrix} 6 & 6 & 0 \\ -3 & -3 & 0 \\ -3 & -6 & 3 \end{pmatrix}\begin{pmatrix} x & y & z \end{pmatrix} = 0$$

o sea,

$$\begin{aligned} 6x + 6y &= 0 \\ -3x - 3y &= 0 \\ -3x - 6y + z &= 0 \end{aligned}$$

so $S(-2) = \{-z, z, z) : z \in \mathbb{R}\} =< (-1, 1, 1) >$.

The matrix $A$ is diagonalizable and $A = PDP^{-1}$ with

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & 2 & -1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

The characteristic polynomial of $B$ is

$$|B - \lambda I| = \begin{pmatrix} 1-\lambda & 0 & -2 \\ 0 & -\lambda & 0 \\ -2 & 0 & 4-\lambda \end{pmatrix} = -\lambda^2(\lambda - 5)$$

so its eigenvalues are $\lambda_1 = 0$ with multiplicity $n_1 = 2$ and $\lambda_2 = 5$ with multiplicity $n_2 = 1$.

We compute $S(0)$ by solving the linear system of equations

$$(b - 0I)\begin{pmatrix} x & y & z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}\begin{pmatrix} x & y & z \end{pmatrix} = 0$$

that is,

$$\begin{aligned} x - 2z &= 0 \\ -2x + 4z &= 0 \end{aligned}$$

We use $y$ and $z$ as parameters. The solution is $x = 2z$. Hence, $S(0) = \{2z, y, z) : y, z \in \mathbb{R}\} =< (2, 0, 1), (0, 1, 0) >$, so $\dim S(0) = 2$.

Now, $S(5)$ is the set of solutions to the system of linear equations

$$(B - 5I)\begin{pmatrix} x & y & z \end{pmatrix} = \begin{pmatrix} -4 & 0 & -2 \\ 0 & -5 & 0 \\ -2 & 0 & -1 \end{pmatrix}\begin{pmatrix} x & y & z \end{pmatrix} = 0$$

that is,

$$\begin{aligned} -4x - 2z &= 0 \\ -5y &= 0 \\ -2x - z &= 0 \end{aligned}$$

Using $x$ as the parameter, we find that $y = 0$, $z = -2x$. Hence, $S(5) = \{x, 0, -2x) : x \in \mathbb{R}\} =< (1, 0, -2) >$.

The matrix $B$ is diagonalizable and $B = QDQ^{-1}$ with

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix}$$

The characteristic polynomial of $C$. is

$$|C - \lambda I| = \begin{pmatrix} 4-\lambda & 5 & -2 \\ -2 & -2-\lambda & 1 \\ -1 & -1 & 1-\lambda \end{pmatrix} = -(\lambda-1)^3$$

so there is only one eigenvalue $\lambda_1 = 1$ with multiplicity $n_1 = 3$. The space $S(1)$ is the set of solutions to the system of linear equations

$$(C - I)\begin{pmatrix} x & y & z \end{pmatrix} = \begin{pmatrix} 3 & 5 & -2 \\ -2 & -3 & 1 \\ -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x & y & z \end{pmatrix} = 0$$

that is,

$$\begin{aligned} 3x + 5y - 2z &= 0 \\ -2x - 3y + z &= 0 \\ -x - y &= 0 \end{aligned}$$

the solution is $y = -x$, $z = -x$. Using $z$ as the parameter $S(0) = \{-z, z, z) : z \in \mathbb{R}\} =< (-1, 1, 1) >$ so $\dim S(1) = 1 < n_1 = 3$ and the matrix $C$ is not diagonalizable.

(3) *What are the values of $a$ for which the matrix*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

*is diagonalizable?*

**Solution:** The characteristic polynomial is

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ a & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda)$$

There are two eigenvalues $\lambda_1 = 1$, with multiplicity 2 and $\lambda_2 = 2$ with multiplicity 1.

The matrix is diagonalizable if and only if $\dim S(1) = 2$. The space $S(1)$is the set of solutions to the system of linear equations

$$(A - \lambda I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

which is the same as

$$\begin{aligned} ax &= 0 \\ x + y + z &= 0 \end{aligned}$$

If $a \neq 0$ the solutions are $x = 0$, $y = -z$. Hence, $S(1) = \{(0, -z, z) : z \in \mathbb{R}\}$ and we see that $\dim S(1) = 1$. Hence, if $a \neq 0$ $A$ is not diagonalizable.

But, if $a = 0$, the system becomes

$$x + y + z = 0$$

so $S(1) = \{(x, y, -x-y) : x, y \in \mathbb{R}\}$ and $\dim S(1) = 2$. In this case, $A$ is diagonalizable.

(4) *Show that*
  (a) *If $A$ is a diagonalizable matrix, so is $A^n$ for each $n \in \mathbb{N}$.*
  (b) *A diagonalizable matrix $A$ is regular if and only if none of its eigenvalues vanishes.*

(c) *If $A$ has an inverse, then both $A$ and $A^{-1}$ have the same eigenvectors and the eigenvalues of $A$ are the reciprocal of the eigenvalues of $A^{-1}$.*
(d) *$A$ and $A^t$ have the same eigenvalues.*

**Solution:** Since, $|A^t - \lambda I| = \left|(A-\lambda I)^t\right| = |A - \lambda I|$ the characteristic polynomials of $A$ and $A^t$ are the same. Therefore, the eigenvalues are the same.

(5) *Study for which values of $a$ and $b$ the matrix $A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & a \\ 3 & 0 & b \end{pmatrix}$ is diagonalizable.*

**Solution:** The characteristic polynomial is

$$|A-\lambda I| = \left|\begin{array}{ccc} 5-\lambda & 0 & 0 \\ 0 & -1-\lambda & a \\ 3 & 0 & b-\lambda \end{array}\right| = (5-\lambda)\left|\begin{array}{cc} -1-\lambda & a \\ 0 & b-\lambda \end{array}\right| = (5-\lambda)(1+\lambda)(b-\lambda)$$

So, the eigenvalues are $\lambda_1 = 5$, $\lambda_2 = -1$ y $\lambda_3 = b$. If $b \neq 5$ y $b \neq -1$ there are three different eigenvalues and the matrix is diagonalizable.

If $b = 5$ then $\lambda_1 = 5$ has multiplicity $n_1 = 2$ and the other eigenvalue has multiplicity 1. The matrix is diagonalizable or not depending on the dimension of $S(5)$. This space is the set of solutions to the system of linear equations

$$(A-5I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -6 & a \\ 3 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

which is the same as

$$\begin{aligned} -6y + az &= 0 \\ 3x &= 0 \end{aligned}$$

Clearly, $\dim S(5) = 1 < n_1 = 2$, so $A$ is not diagonalizable.

On the other hand, if $b = -1$ the eigenvalues are $\lambda_1 = 5$, with multiplicity $n_1 = 1$ and $\lambda_2 = -1$ with multiplicity $n_2 = 2$. Now The matrix is diagonalizable or not depending on the dimension of $S(-1)$. This space is the set of solutions to the system of linear equations

$$(A+I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & a \\ 3 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

that is,

$$\begin{aligned} 6x &= 0 \\ az &= 0 \\ 3x &= 0 \end{aligned}$$

and we see that

$$\dim S(-1) = \begin{cases} 1, & \text{si } a \neq 0; \\ 2, & \text{si } a = 0. \end{cases}$$

We could have done this in an easier way, by noting that

$$\dim S(-1) = \operatorname{rg}(A+I) = \operatorname{rg}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & a \\ 3 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \operatorname{rg}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{cases} 1, & a \neq 0; \\ 2, & \text{si } a = 0. \end{cases}$$

Thus, if

$$\begin{cases} b = 5, & \text{then } A \text{ is not diagonalizable;} \\ b = -1, & \text{then } A \text{ is diagonalizable and only if } a = 0; \\ b \neq 5 \text{ y } b \neq -1, & \text{then } A \text{ is diagonalizable.} \end{cases}$$

(6) *Which of the following matrices are diagonalizable?*

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \qquad C \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

**Solution:** The characteristic polynomial of $A$ is

$$(A - \lambda I) = \begin{pmatrix} 1-\lambda & 2 & 0 \\ -1 & 3-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{pmatrix} = -(\lambda - 1)(\lambda - 2)^2$$

so the eigenvalues are $\lambda_1 = 1$ with multiplicity $n_1 = 1$ and $\lambda_2 = 2$ with multiplicity $n_2 = 2$.

The space $S(2)$ is the set of solutions to the system of linear equations

$$(A - 2I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

that is,

$$\begin{aligned} -x + 2y \quad &= 0 \\ -x + y + z \quad &= 0 \\ y - z \quad &= 0 \end{aligned}$$

The solution is $y = z$, $x = 2y = 2z$ ($z$ is the parameter). Hence, $S(2) = \{(2z, z, z) : z \in \mathbb{R}\} =< (2,1,1) >$ so $\dim S(2) = 1 < n_2 = 2$ and $A$ is not diagonalizable.

The characteristic polynomial of $B$ is

$$|B - \lambda I| = \begin{vmatrix} -2-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 2\lambda - 1$$

so the eigenvalues are

$$\begin{aligned} \lambda_1 &= \tfrac{-2+\sqrt{4+4}}{2} = -1 + \sqrt{2} \\ \lambda_2 &= \tfrac{-2-\sqrt{4+4}}{2} = -1 - \sqrt{2} \end{aligned}$$

all of the with multiplicity 1. Hence, $B$ is diagonalizable.

The space $S(-1+\sqrt{2})$ is the set of solutions to the system of linear equations

$$\left(A - (-1+\sqrt{2})I\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1-\sqrt{2} & 1 \\ 1 & 1-\sqrt{2} \end{pmatrix}\begin{pmatrix} x \\ y \\ = 0 \end{pmatrix}$$

that is,

$$\begin{aligned} -(1+\sqrt{2})x + y \quad &= 0 \\ x + (1-\sqrt{2})y \quad &= 0 \end{aligned}$$

the solution is $x = y/(1+\sqrt{2})$. Hence, $S(-1+\sqrt{2}) = \{(y/(1+\sqrt{2}), y) : y \in \mathbb{R}\} =< (1/(1+\sqrt{2}), 1) >=< (1, 1+\sqrt{2}) >$. Likewise, $S(-1+\sqrt{2}) = \{(y/(1-\sqrt{2}), y) : y \in \mathbb{R}\} =< (1/(1-\sqrt{2}), 1) >=< (1, 1-\sqrt{2}) >$.

The diagonal form of $B$ is $B = PDP^{-1}$ with

$$P = \begin{pmatrix} 1 & 1 \\ 1+\sqrt{2} & 1-\sqrt{2} \end{pmatrix} \qquad D = \begin{pmatrix} -1+\sqrt{2} & 0 \\ 0 & -1-\sqrt{2} \end{pmatrix}$$

Finally, the characteristic polynomial of $C$ is

$$|C - \lambda I| = \begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = (\lambda - 1)^2$$

so there is a unique eigenvalue $\lambda = 1$ with multiplicity 2. The eigenspace $S(1)$ is the set of solutions to the system of linear equations

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

so $y = 0$. Hence, $S(1) = \{(x, 0) : x \in \mathbb{R}\} =< (1, 0) >$ and $\dim S(1) = 1$. Therefore, the $C$ is not diagonalizable.

(7) *The matrix* $\begin{pmatrix} 1 & 0 & 0 \\ \alpha+1 & 2 & 0 \\ 0 & \alpha+1 & 1 \end{pmatrix}$ *is diagonalizable if and only $\alpha$ is...*

**Solution:** The characteristic polynomial of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ \alpha+1 & 2 & 0 \\ 0 & \alpha+1 & 1 \end{pmatrix}$$

is $(\lambda-1)^2(\lambda-2)$. The eigenvalues are $\lambda_1 = 1$ with multiplicity $n_1 = 2$ and $\lambda_2 = 2$ with multiplicity $n_2 = 1$. The matrix $A$ is diagonalizable if and only if $\dim S(1) = 2$. The subspace $S(1)$ is the set of solutions to the system of linear equations

$$\left.\begin{array}{ll} (\alpha+1)x + y & = 0 \\ (\alpha+1)y & = 0 \end{array}\right\}$$

If $\alpha \neq -1$ the solution is $x = y = 0$. That is, $S(1) = \{(0, 0, z) : z \in \mathbb{R}\}$ and $\dim S(1) = 1$. Therefore, if $\alpha \neq -1$ then $A$ is not diagonalizable.

If $\alpha = -1$ the linear system above reduces to $y = 0$.In this case, $S(1) = \{(x, 0, z) : x, z \in \mathbb{R}\}$ and $\dim S(1) = 2$. So, if $\alpha = -1$ the matrix $A$ is diagonalizable.

(8) *Consider the matrices*

$$A = \begin{pmatrix} 3 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

*Find whether they are diagonalizable and, whenever they are, compute their $n$-th power.*

**Solution:** Let

$$A = \begin{pmatrix} 3 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The eigenvalues are $\lambda_1 = 1$ with multiplicity $n_1 = 2$ and $\lambda_2 = 2$ with multiplicity $n_2 = 1$. The eigenspaces are $S(1) =< (0, 0, 1), (-1, 1, 0) >$ y $S(2) =< (-2, 1, 0) >$. The matrix $A$ is diagonalizable $A = PDP^{-1}$ with

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & -1 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

So,

$$A^n = \begin{pmatrix} 0 & -1 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix}\begin{pmatrix} 0 & -1 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} 0 & -1 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ -1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} -1+2^{1+n} & -2+2^{1+n} & 0 \\ 1-2^n & 2-2^n & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

The eigenvalues are $\lambda_1 = 0$ with multiplicity $n_1 = 1$ and $\lambda_2 = 2$ with multiplicity $n_2 = 2$. The eigenspaces are $S(0) =< (0,1,1) >$ and $S(2) =< (1,0,1),(1,1,0) >$. The matrix $B$ is diagonalizable: $B = PDP^{-1}$ with

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

so,

$$B^n = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1}$$

$$= \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 2^n & 0 & 0 \\ 2^{n-1} & 2^{n-1} & -2^{n-1} \\ 2^{n-1} & -2^{n-1} & 2^{n-1} \end{pmatrix}$$

The eigenvalues of

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

are $\lambda_1 = 1$ with multiplicity $n_1 = 2$ and $\lambda_2 = 2$ with multiplicity $n_2 = 1$. The eigenspaces are $S(1) =< (1,0,0),(0,1,0) >$ and $S(2) =< (0,1,1) >$. The matrix $C$ is diagonalizable: $C = PDP^{-1}$ with

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

so,

$$C^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2^n - 1 \\ 0 & 0 & 2^n \end{pmatrix}$$

(9) *The following are the characteristic polynomials of some square matrices. Determine which of them correspond to diagonalizable matrices.*

| | |
|---|---|
| $p(\lambda) = \lambda^2 + 1$ | $p(\lambda) = \lambda^2 - 1$ |
| $p(\lambda) = \lambda^2 + \alpha$ | $p(\lambda) = \lambda^2 + 2\alpha\lambda + 1$ |
| $p(\lambda) = \lambda^2 + 2\lambda + 1$ | $p(\lambda) = (\lambda - 1)^3$ |
| $p(\lambda) = \lambda^3 - 1$ | |

**Solution:**

**1)** $p(\lambda) = \lambda^2 + 1$. The matrix is not diagonalizable because not all the roots are real numbers.
**2)** $p(\lambda) = \lambda^2 + \alpha$. If $\alpha > 0$ the matrix is no diagonalizable because not all the roots are real numbers. If $\alpha < 0$ the characteristic polynomial has two different real roots, so the matrix is diagonalizable. If $\alpha = 0$ there is a unique eigenvalue 0 with multiplicity 2. Hence, either all the entries in the matrix are 0, or else the matrix is no diagonalizable.

**3)** $p(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2$. We see that $-1$ is a double root. Therefore, either the matrix is $-I$ , or else the matrix is no diagonalizable.
**4)** $p(\lambda) = \lambda^3 - 1 = (\lambda-1)(\lambda^2+\lambda+1)$ has no real roots. The matrix is no diagonalizable.
**5)** $p(\lambda) = \lambda^2 - 1$ has two distinct real roots. The matrix is diagonalizable.
**6)** $p(\lambda) = \lambda^2 + 2\alpha\lambda + 1$. The roots are $\lambda = -\alpha \pm \sqrt{\alpha^2 - 1}$. Thus,

- If $|\alpha| > 1$, the matrix is diagonalizable.
- If $|\alpha| < 1$, the matrix is not diagonalizable.
- If $|\alpha| = 1$, we are in case 3).

(10) *Determine whether the following matrices are diagonalizable. Compute the $n$-th power whenever they are diagonalizable.*

$$A = \begin{pmatrix} \alpha & 0 \\ 1 & \alpha \end{pmatrix} \quad B = \begin{pmatrix} \alpha & 1 \\ 1 & \alpha \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

**Solution:**
**1)** The matrix $A$ is of order 2 and its unique eigenvalue is $\alpha$ of multiplicity 2. Therefore, $A$ is not diagonalizable.
**2)** The characteristic polynomial of $B$ is $(\lambda - \alpha)^2 - 1$. The roots are $\alpha \pm 1$ so $B$ is diagonalizable. The eigenvalues are

$$S(\alpha - 1) =< (-1, 1) >, \qquad S(\alpha + 1) =< (1, 1) >$$

and $B = PDP^{-1}$ con

$$P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} \alpha - 1 & 0 \\ 0 & \alpha + 1 \end{pmatrix}$$

Thus,

$$B^n = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (\alpha-1)^n & 0 \\ 0 & (\alpha+1)^n \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1} =$$

$$\frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (\alpha-1)^n & 0 \\ 0 & (\alpha+1)^n \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (\alpha-1)^n + (\alpha+1)^n & -(\alpha-1)^n + (\alpha+1)^n \\ -(\alpha-1)^n + (\alpha+1)^n & (\alpha-1)^n + (\alpha+1)^n \end{pmatrix}$$

**3)** The eigenvalues of

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

are $\lambda_1 = 1$ with multiplicity $n_1 = 3$. Since,

$$S(1) =< (1, 0, 0) >$$

the matrix is not diagonalizable.

(11) *Study for what values of the parameters the following matrices are diagonalizable. Find the eigenvalues and eigenvectors.*

$$A = \begin{pmatrix} a & b & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & -2-\alpha \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix}$$

(12) *The matrix*

$$\begin{pmatrix} a & 1 & p \\ b & 2 & q \\ c & -1 & r \end{pmatrix}$$

*has $(1, 1, 0)$, $(-1, 0, 2)$ and $(0, 1, -1)$ as eigenvectors. Compute its eigenvalues.*

(13) *Determine whether the following matrices are diagonalizable. If possible, write their diagonal form.*

$$A = \begin{pmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 5 & 7 & 5 \\ -6 & -5 & -3 \\ 4 & 1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 2 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad E = \begin{pmatrix} -1 & 2 & -2 \\ 0 & 2 & 0 \\ 0 & 3 & -2 \end{pmatrix} \quad F = \begin{pmatrix} 5 & -10 & 8 \\ -10 & 2 & 2 \\ 8 & 2 & 11 \end{pmatrix}$$

$$G = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 2 \\ 0 & 1 & 4 \end{pmatrix} \quad H = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ -1 & 0 & -2 \end{pmatrix} \quad I = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \quad K = \begin{pmatrix} -1 & 2 & -2 \\ 0 & 2 & 0 \\ 0 & 3 & -2 \end{pmatrix} \quad L = \begin{pmatrix} -9 & 1 & 1 \\ -18 & 0 & 3 \\ -21 & 4 & 0 \end{pmatrix}$$

**Solution:**
**1)** The eigenvalues of

$$A = \begin{pmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix}$$

are $-2, 4, 4$. Also, $S(-2) =< (-1, 1, 1) >$, $S(4) =< (1, -1, 1) >$, so the matrix is diagonalizable.
**2)** The eigenvalues of

$$B = \begin{pmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

are $-1, -1, -1$. Since, $B$ is not already in diagonal form, it is not diagonalizable.
**5)** The eigenvalues of

$$E = \begin{pmatrix} -1 & 2 & -2 \\ 0 & 2 & 0 \\ 0 & 3 & -2 \end{pmatrix}$$

are $-2, -1, 2$. Since they are all distinct then $E$ is diagonalizable. Also, $E = PDP^{-1}$ with

$$P = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 12 \\ 1 & 0 & 9 \end{pmatrix} \qquad D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

**12)** The eigenvalues of

$$L = \begin{pmatrix} -9 & 1 & 1 \\ -18 & 0 & 3 \\ -21 & 4 & 0 \end{pmatrix}$$

are $-3, -3, -3$. Since $L$ is not already in diagonal form, it is not diagonalizable.

(14) *For what values of the parameter $a$ is the quadratic form $Q(x, y, z) = x^2 - 2axy - 2xz + y^2 + 4yz + 5z^2$ positive definite?*

**Solution:** $Q(x, y, z) = x^2 - 2axy - 2xz + y^2 + 4yz + 5z^2$
It will be positive definite if $D_1 > 0, D_2 > 0, D_3 > 0$. Let us compute these.
$D_1 = 1$

$D_2 = \begin{vmatrix} 1 & -a \\ -a & 1 \end{vmatrix} = 1 - a^2 > 0$ if and only if $|a| < 1$.

$D_3 = \begin{vmatrix} 1 & -a & -1 \\ -a & 1 & 2 \\ -1 & 2 & 5 \end{vmatrix} = -5a^2 + 4a = a(4-5a) > 0$ if and only if $a \in (0, 4/5)$.

Therefore, the quadratic form is positive definite if $a \in (0, 4/5)$. When $a = 0$ or $a = 4/5$, we have that $D_1 > 0$, $D_2 > 0$, $D_3 = 0$. So, the quadratic form is positive semidefinite, but not positive definite. When $a \in (-\infty, 0) \cup (\frac{4}{5}, +\infty)$ we see that $D_1 > 0$, $D_3 < 0$ so the quadratic form is indefinite.

(15) *Study the signature of the following quadratic forms.*
(a) $Q_1(x,y,z) = x^2 + 7y^2 + 8z^2 - 6xy + 4xz - 10yz$.
(b) $Q_2(x,y,z) = -2y^2 - z^2 + 2xy + 2xz + 4yz$.

**Solution:** a) The matrix associated to $Q_1$ is $\begin{pmatrix} 1 & -3 & 2 \\ -3 & 7 & -5 \\ 2 & -5 & 8 \end{pmatrix}$. Let us compute $D_1 = 1 > 0$, $D_2 = \begin{vmatrix} 1 & -3 \\ -3 & 7 \end{vmatrix} = -2$ and $D_3 = \begin{vmatrix} 1 & -3 & 2 \\ -3 & 7 & -5 \\ 2 & -5 & 8 \end{vmatrix} = -9$. Therefore, the quadratic form is indefinite. (Note that it was not necessary to compute $D_3$)

b) The matrix associated to $Q_2$ is $\begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & 2 & -1 \end{pmatrix}$. We see that $D_1 = 0$. Can we still apply the method of principal minors? To do so we perform the following change of variables: $\bar{x} = z$, $\bar{z} = x$. We see that

$$Q_2(\bar{x}, y, \bar{z}) = -2y^2 - \bar{x}^2 + 2\bar{z}y + 2\bar{x}\bar{z} + 4y\bar{x}$$

whose associated matrix is $\begin{pmatrix} -1 & 2 & 1 \\ 2 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. The principal minors are $D_1 = -1$, $D_2 = \begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix} = -2$. Therefore, the quadratic form is indefinite.

Here is another way to do this exercise. Since, $D_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & 2 & -1 \end{vmatrix} = 7 \neq 0$. But, $D_1 = 0$, $D_2 = -1$, so by Proposition 3.13, the quadratic form is indefinite.

(16) *Study for what values of $a$ the quadratic form $Q(x,y,z) = ax^2 + 4ay^2 + 4az^2 + 4xy + 2axz + 4yz$ is*
(a) *positive definite.*
(b) *negative definite.*

**Solution:** The matrix associated to the quadratic form $Q(x,y,z) = ax^2 + 4ay^2 + 4az^2 + 4xy + 2axz + 4yz$ is

$$\begin{pmatrix} a & 2 & a \\ 2 & 4a & 2 \\ a & 2 & 4a \end{pmatrix}$$

(a) We study conditions under which the principal minors satisfy the following
(i) $D_1 = a > 0$.

(ii) $D_2 = \begin{vmatrix} a & 2 \\ 2 & 4a \end{vmatrix} = 4a^2 - 4 = 4(a^2-1) > 0$. This condition is satisfied if and only if $|a| > 1$

(iii) $D_3 = \begin{vmatrix} a & 2 & a \\ 2 & 4a & 2 \\ a & 2 & 4a \end{vmatrix} = 12a^3 - 12a = 12a(a^2-1) > 0.$

Assuming $a > 0$, the condition $a(a^2-1) > 0$ simplifies to $(a^2-1) > 0$ which is satisfied if and only if $|a| > 1$. Therefore, $Q$ es positive definite if $a > 1$.

(b) We study conditions under which the principal minors satisfy the following

(i) $D_1 = a < 0$.

(ii) $D_2 = \begin{vmatrix} a & 2 \\ 2 & 4a \end{vmatrix} = 4a^2 - 4 = 4(a^2-1) > 0$ This condition is satisfied if and only if $|a| > 1$.

Assuming, $a < 0$, the equation $4(a^2-1) > 0$ implies that $a < -1$. In the previous part we have seen that $D_3 = 12a(a^2-1) < 0$ if $a < -1$. Therefore, $Q$ is definite negative if $a < -1$.

The above remarks show that $Q$ is indefinite if $a \in (-1,0) \cup (0,1)$. If $a = 0$, the quadratic form is $Q(x,y,z) = 4xy + 4yz$ and we see that $Q(1,1,0) = 4 > 0$, $Q(1,-1,0) = -4 < 0$, so $Q$ is indefinite.

To study the cases $a = \pm 1$ we do the following change of variables

$$\bar{x} = z, \quad \bar{y} = y, \quad \bar{z} = x$$

and we obtain the quadratic form

$$Q(\bar{x},\bar{y},\bar{z}) = a\bar{z}^2 + 4a\bar{y}^2 + 4a\bar{x}^2 + 4\bar{z}\bar{y} + 2a\bar{z}\bar{x} + 4\bar{y}\bar{x} = 4a\bar{x}^2 + 4a\bar{y}^2 + a\bar{z}^2 + 4\bar{x}\bar{y} + 2a\bar{z}\bar{x} + 4\bar{y}\bar{x}$$

whose associated matrix is

$$\begin{pmatrix} 4a & 2 & a \\ 2 & 4a & 2 \\ a & 2 & a \end{pmatrix}$$

For this matrix we see that that

$$D_1 = 4a, D_2 = 16a^2 - 4, \quad D_3 = 12a(a^2-1)$$

And, for $a = 1$ we obtain that

$$D_1 = 4, D_2 = 8, \quad D_3 = 0$$

so $Q$ is positive semidefinite. Finally, for $a = -1$ we obtain that

$$D_1 = -4, D_2 = 8, \quad D_3 = 0$$

so $Q$ is negative semidefinite.

(17) *Classify the following quadratic forms, depending on the parameters.*

a) $Q(x,y,z) = 9x^2 + 3y^2 + z^2 + 2axz$

b) $Q(x_1,x_2,x_3) = x_1^2 + 4x_2^2 + bx_3^2 + 2ax_1x_2 + 2x_2x_3$

**Solution:** a) The matrix associated to $Q(x,y,z) = 9x^2+3y^2+z^2+2axz$ is $\begin{pmatrix} 9 & 0 & a \\ 0 & 3 & 0 \\ a & 0 & 1 \end{pmatrix}$.

The principal minors are $D_1 = 9$, $D_2 = \begin{vmatrix} 9 & 0 \\ 0 & 3 \end{vmatrix} = 27$ y $D_3 = \begin{vmatrix} 9 & 0 & a \\ 0 & 3 & 0 \\ a & 0 & 1 \end{vmatrix} = 27 - 3a^2$.

Therefore, the quadratic form is

(a) definite positive if $27 - 3a^2 > 0$ that is if, $-3 < a < 3$.

(b) cannot be negative definite since $D_1 = 9 > 0$.

(c) cannot be negative semidefinite either.
(d) is positive semidefinite if $27 - 3a^2 = 0$. That is, if $a = -3$ $a = 3$.
(e) is indefinite if $27 - 3a^2 < 0$. That is, if $|a| > 3$.

b) The matrix associated to $Q(x_1, x_2, x_3) = x_1^2 + 4x_2^2 + bx_3^2 + 2ax_1x_2 + 2x_2x_3$ is $\begin{pmatrix} 1 & a & 0 \\ a & 4 & 1 \\ 0 & 1 & b \end{pmatrix}$.

The principal minors are $D_1 = 1 > 0$, $D_2 = \begin{vmatrix} 1 & a \\ a & 4 \end{vmatrix} = 4 - a^2$ y $D_3 = \begin{vmatrix} 1 & a & 0 \\ a & 4 & 1 \\ 0 & 1 & b \end{vmatrix} = 4b - 1 - a^2b = b(4 - a^2) - 1$.

Hence,

(a) the quadratic form is positive definite if

$$\left.\begin{array}{c} 4 - a^2 > 0 \\ 4b - 1 - a^2b > 0 \end{array}\right\}$$

From the first inequality we obtain the condition $-2 < a < 2$. De la segunda $b > \frac{1}{4-a^2}$. That is, if

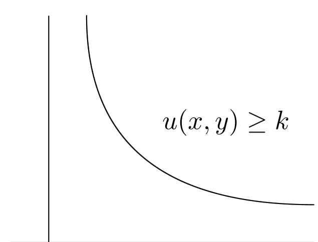
$$\left.\begin{array}{c} -2 < a < 2 \\ b > \frac{1}{4-a^2} \end{array}\right\}$$

(b) the quadratic form cannot be negative definite or semidefinite because $D_1 = 1 > 0$
(c) If $a \in (-2, 2)$ y $b = \frac{1}{4-a^2}$, then $D_3 = 4b - 1 - a^2b = 0$ so the quadratic form is positive semidefinite.
(d) If $|a| > 2$ (so, $4 - a^2 < 0$), then the quadratic form is indefinite.
(e) Finally, if $|a| = 2$, we get that $\begin{pmatrix} 1 & a & 0 \\ a & 4 & 1 \\ 0 & 1 & b \end{pmatrix}$. The principal minors are

$$D_1 = 1, \quad D_2 = 4 - a^2 = 0, \quad D_3 = 4b - 1 - a^2b = -1$$

and the quadratic form is indefinite.

(18) *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a concave function so that for every $v_1, v_2 \in \mathbb{R}^n$ and $\lambda \in [0, 1]$, we have that $u(\lambda v_1 + (1 - \lambda)v_2) \geq \lambda u(v_1) + (1 - \lambda)u(v_2)$. Show that $S = \{v \in \mathbb{R}^n : u(v) \geq k\}$ is a convex set. For a concave $u : \mathbb{R}^2 \to \mathbb{R}$, the figure represents its graph $S = \{(x, y) \in \mathbb{R}^2 : u(x, y) \geq k\}$*




**Solution:** Let $S = \{x \in \mathbb{R}^n : u(x) \geq k\}$. Let $x, y \in S$, so $u(x) \geq k$ and also $u(y) \geq k$. Given a convex combination of these two points, $x_c = \lambda x + (1 - \lambda)y$ we have that

$$\begin{aligned} u(x_c) = & u(\lambda x + (1 - \lambda)y) \\ & \geq \lambda u(x) + (1 - \lambda)u(y) \qquad \text{since } u \text{ is concave} \\ & \geq \lambda k + (1 - \lambda)k = k \end{aligned}$$

Therefore, $x_c \in S$ and $S$ is convex.

(19) *State the previous problem for a convex function $u : \mathbb{R}^n \to \mathbb{R}$.*

**Solution:** Let $u : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then, the set $S = \{x \in \mathbb{R}^n : u(x) \leq k\}$ is convex.

(20) *Determine the domains of the plane where the following functions are convex or concave.*
(a) $f(x,y) = (x-1)^2 + xy^2$.
(b) $g(x,y) = \frac{x^3}{3} - 4xy + 12x + y^2$.
(c) $h(x,y) = e^{-x} + e^{-y}$.
(d) $k(x,y) = e^{xy}$.
(e) $l(x,y) = \ln\sqrt{xy}$.

**Solution:**
(a) First, note that if $x = 0$ then $f(0,y) = 1$ is constant. Hence, $f$ is concave and convex in the set $\{(0,y) : y \in \mathbb{R}\}$. The Hessian matrix of $f(x,y) = (x-1)^2 + xy^2$ is

$$\begin{pmatrix} 2 & 2y \\ 2y & 2x \end{pmatrix}$$

We see that $D_1 = 2 > 0$, $D_2 = 4(x - y^2)$. Since, $D_1 > 0$ the function is not concave in any non-empty subset of $\mathbb{R}^2$. We see that $D_2 \geq 0$ if and only if $x \geq y^2$. The function is convex in the set $\{(x,y) \in \mathbb{R}^2 : x \geq y^2\}$.
(b) The Hessian matrix of

$$f(x,y) = \frac{x^3}{3} - 4xy + 12x + y^2$$

is

$$\begin{pmatrix} 2x & -4 \\ -4 & 2 \end{pmatrix}$$

We see that $D_1 = 2x$, $D_2 = 4x - 16$. The function is concave in the convex sets in which $D_1 < 0$ (so $x < 0$) and $D_2 \geq 0$ (that is, $x \geq 4$). Since, both conditions are not compatible, the function is not concave in any non-empty set of $\mathbb{R}^2$.
If $x > 0$ y $x \geq 4$ then $D_1 > 0$ y $D_2 \geq 0$ and we see that the function is convex in the set $\{(x,y) \in \mathbb{R}^2 : x \geq 4\}$.
(c) The Hessian matrix of $h(x,y) = e^{-x} + e^{-y}$ is

$$\begin{pmatrix} e^{-x} & 0 \\ 0 & e^{-y} \end{pmatrix}$$

Both second derivatives are positive. Hence, the function is convex in $\mathbb{R}^2$.
(d) The Hessian matrix of $k(x,y) = e^{xy}$ is

$$e^{yx}\begin{pmatrix} y^2 & xy+1 \\ xy+1 & x^2 \end{pmatrix}$$

Since, $e^{yx} > 0$ for every $(x,y) \in \mathbb{R}^2$, the signature of the above matrix is the same as the signature of the following one

$$\begin{pmatrix} y^2 & xy+1 \\ xy+1 & x^2 \end{pmatrix}$$

For this matrix we obtain that $D_1 = y^2 \geq 0$, $D_2 = -1 - 2xy$. The function is convex if $D_2 > 0$. That is, if $2xy < -1$. Therefore, the function is convex in the set

$$A = \{(x,y) \in \mathbb{R}^2 : xy < -1/2, x > 0\}$$

and also in the set

$$B = \{(x,y) \in \mathbb{R}^2 : xy < -1/2, x < 0\}$$

The union $A \cup B$ is not a convex set. Finally, in the convex sets $C = \{(x,y) \in \mathbb{R}^2 : x = 0\}$ and $D = \{(x,y) \in \mathbb{R}^2 : y = 0\}$ the function is constant and hence, both concave and convex.

(e) The Hessian matrix of

$$l(x,y)=\ln(\sqrt{xy})=\begin{cases}\frac{1}{2}(\ln x+\ln y), & \text{if } x,y>0;\\ \frac{1}{2}(\ln(-x)+\ln(-y)), & \text{if } x,y<0;\end{cases}$$

is

$$\frac{1}{2}\begin{pmatrix}-\frac{1}{x^2} & 0\\ 0 & -\frac{1}{y^2}\end{pmatrix}$$

Clearly, this matrix is negative definite and, therefore, function is concave in $\mathbb{R}^2_{++}$ and in $\mathbb{R}^2_{--}$.

(21) *Determine the values of the parameters $a$ and $b$ so that the following functions are convex in their domains.*
(a) $f(x,y,z)=ax^2+y^2+2z^2-4axy+2yz$
(b) $g(x,y)=4ax^2+8xy+by^2$

**Solution:**
(a) The Hessian of $f(x,y,z)=ax^2+y^2+2z^2-4axy+2yz$ is

$$\begin{pmatrix}2a & -4a & 0\\ -4a & 2 & 2\\ 0 & 2 & 4\end{pmatrix}$$

Note that

$$\begin{aligned}D_1 &= 2a\\ D_2 &= \begin{vmatrix}2a & -4a\\ -4a & 2\end{vmatrix}=4a-16a^2=4a(1-4a)\\ D_3 &= \begin{vmatrix}2a & -4a & 0\\ -4a & 2 & 2\\ 0 & 2 & 4\end{vmatrix}=8a-64a^2=8a(1-8a)\end{aligned}$$

Thus, $D_1>0$ is equivalent to $a>0$. Assuming this, the condition $D_3>0$ is equivalent to $a<1/8$. Furthermore, if $0<a<1/8$ then $D_2>0$, so the function is strictly convex if $0<a<1/8$. On the other hand, if $a=0$ or $a=1/8$, the Hessian positive semidefinite. Therefore, the function is convex if $0\le a\le 1/8$.

(b) The Hessian of $g(x,y)=4ax^2+8xy+by^2$ is

$$\begin{pmatrix}8a & 8\\ 8 & 2b\end{pmatrix}$$

Note that

$$\begin{aligned}D_1 &= 8a\\ D_2 &= \begin{vmatrix}8a & 8\\ 8 & 2b\end{vmatrix}=16(ab-4)\end{aligned}$$

The function is convex if $a>0$ and $ab\ge 4$. This is equivalent to $a>0$ and $b\ge 4/a$.
If $a=0$, then $D_1=0$, $D_2=-64\ne 0$. Hence, $\mathrm{H}\,h(x,y)$ is indefinite for every $(x,y)\in\mathbb{R}^2$ and the function is not convex in $\mathbb{R}^2$.
If $a<0$, then $D_1<0$, so $\mathrm{H}\,h(x,y)$ cannot be positive definite or positive semidefinite at any $(x,y)\in\mathbb{R}^2$ and the function is not convex in $\mathbb{R}^2$.

(22) *Discuss the concavity and convexity of the function $f(x,y)=-6x^2+(2a+4)xy-y^2+4ay$ according to the values of $a$.*

**Solution:** The Hessian of $f(x,y) = -6x^2 + (2a+4)xy - y^2 + 4ay$ is

$$\begin{pmatrix} -12 & 2a+4 \\ 2a+4 & -2 \end{pmatrix}$$

We have that

$$\begin{aligned} D_1 =& -12 < 0 \\ D_2 =& \begin{vmatrix} -12 & 2a+4 \\ 2a+4 & -2 \end{vmatrix} = 8 - 4a^2 - 16a \end{aligned}$$

Since $D_1 < 0$ the function cannot be convex. It would be concave if $D_2 = 8 - 4a^2 - 16a \geq 0$. The roots of $8 - 4a^2 - 16a = 0$ are $-2 \pm \sqrt{6}$. Thus, $D_2 \geq 0$ is equivalent to $-2 - \sqrt{6} \leq a \leq -2 + \sqrt{6}$. Therefore $f$ is concave if $a \in [-2 - \sqrt{6}, -2 + \sqrt{6}]$.

(23) *Find the largest convex set of the plane where the function* $f(x,y) = x^2 - y^2 - xy - x^3$ *is concave.*

**Solution:** The Hessian of $f(x,y) = x^2 - y^2 - xy - x^3$ is

$$\begin{pmatrix} 2-6x & -1 \\ -1 & -2 \end{pmatrix}$$

We have that

$$\begin{aligned} D_1 =& 2 - 6x \\ D_2 =& 12x - 5 \end{aligned}$$

The condition $D_2 \geq 0$ is equivalent to $x \geq 5/12$. Since $5/12 > 1/3$, the previous inequality also guarantees that $D_1 < 0$. Therefore, the largest set of the plane in which $f$ is concave is the set $\{(x,y) \in \mathbb{R}^2 : x \geq 5/12\}$.