January 26, 2019

MATHEMATICS FOR ECONOMICS II (2018-19)

ECONOMICS, LAW–ECONOMICS, INTERNATIONAL STUDIES–ECONOMICS

SHEET 2. DIAGONALIZATION AND QUADRATIC FORMS

(1) Given the matrix
$$A = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$$
compute its eigenvalues, eigenvectors and diagonalize A.

(2) Given the following matrices
$$A = \begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix} \qquad C = \begin{pmatrix} 4 & 5 & -2 \\ -2 & -2 & 1 \\ -1 & -1 & 1 \end{pmatrix}$$
(a) Compute its eigenvalues, eigenvectors and the eigenspaces.
(b) Diagonalize them, whenever possible.

(3) What are the values of a for which the matrix
$$A = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$
is diagonalizable?

(4) Show that
(a) If A is a diagonalizable matrix, so is A^n for each $n \in \mathbb{N}$.
(b) A diagonalizable matrix A is regular if and only if none of its eigenvalues vanishes.
(c) If A has an inverse, then both A and A^{-1} have the same eigenvectors and the eigenvalues of A are the reciprocal of the eigenvalues of A^{-1}.
(d) A and A^t have the same eigenvalues.

(5) Study for which values of a and b the matrix $A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & a \\ 3 & 0 & b \end{pmatrix}$ is diagonalizable.

(6) Which of the following matrices are diagonalizable?
$$A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \qquad C \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(7) The matrix $\begin{pmatrix} 1 & 0 & 0 \\ \alpha+1 & 2 & 0 \\ 0 & \alpha+1 & 1 \end{pmatrix}$ is diagonalizable if and only α is...

(8) Consider the matrices
$$A = \begin{pmatrix} 3 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$
Find whether they are diagonalizable and, whenever they are, compute their n-th power.

(9) The following are the characteristic polynomials of some square matrices. Determine which of them correspond to diagonalizable matrices.

$p(\lambda) = \lambda^2 + 1$ $\quad p(\lambda) = \lambda^2 - 1$
$p(\lambda) = \lambda^2 + \alpha$ $\quad p(\lambda) = \lambda^2 + 2\alpha\lambda + 1$
$p(\lambda) = \lambda^2 + 2\lambda + 1$ $\quad p(\lambda) = (\lambda - 1)^3$
$p(\lambda) = \lambda^3 - 1$

(10) Determine whether the following matrices are diagonalizable. Compute the n-th power whenever they are diagonalizable.

$$A = \begin{pmatrix} \alpha & 0 \\ 1 & \alpha \end{pmatrix} \quad B = \begin{pmatrix} \alpha & 1 \\ 1 & \alpha \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

(11) Study for what values of the parameters the following matrices are diagonalizable. Find the eigenvalues and eigenvectors.

$$A = \begin{pmatrix} a & b & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & -2-\alpha \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix}$$

(12) The matrix

$$\begin{pmatrix} a & 1 & p \\ b & 2 & q \\ c & -1 & r \end{pmatrix}$$

has $(1,1,0)$, $(-1,0,2)$ and $(0,1,-1)$ as eigenvectors. Compute its eigenvalues.

(13) Determine whether the following matrices are diagonalizable. If possible, write their diagonal form.

$$A = \begin{pmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 5 & 7 & 5 \\ -6 & -5 & -3 \\ 4 & 1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 2 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad E = \begin{pmatrix} -1 & 2 & -2 \\ 0 & 2 & 0 \\ 0 & 3 & -2 \end{pmatrix} \quad F = \begin{pmatrix} 5 & -10 & 8 \\ -10 & 2 & 2 \\ 8 & 2 & 11 \end{pmatrix}$$

$$G = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 2 \\ 0 & 1 & 4 \end{pmatrix} \quad H = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ -1 & 0 & -2 \end{pmatrix} \quad I = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \quad K = \begin{pmatrix} -1 & 2 & -2 \\ 0 & 2 & 0 \\ 0 & 3 & -2 \end{pmatrix} \quad L = \begin{pmatrix} -9 & 1 & 1 \\ -18 & 0 & 3 \\ -21 & 4 & 0 \end{pmatrix}$$

(14) For what values of the parameter a is the quadratic form $Q(x,y,z) = x^2 - 2axy - 2xz + y^2 + 4yz + 5z^2$ positive definite?

(15) Study the signature of the following quadratic forms.
(a) $Q_1(x,y,z) = x^2 + 7y^2 + 8z^2 - 6xy + 4xz - 10yz$.
(b) $Q_2(x,y,z) = -2y^2 - z^2 + 2xy + 2xz + 4yz$.

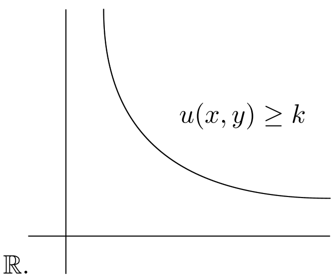
(16) Study for what values of a the quadratic form $Q(x,y,z) = ax^2 + 4ay^2 + 4az^2 + 4xy + 2axz + 4yz$ is

(a) positive definite.
(b) negative definite.

(17) Classify the following quadratic forms, depending on the parameters.

$$\text{a) } Q(x,y,z) = 9x^2 + 3y^2 + z^2 + 2axz$$
$$\text{b) } Q(x_1,x_2,x_3) = x_1^2 + 4x_2^2 + bx_3^2 + 2ax_1x_2 + 2x_2x_3$$

(18) Let $u : \mathbb{R}^n \to \mathbb{R}$ be a concave function so that for every $v_1, v_2 \in \mathbb{R}^n$ and $\lambda \in [0,1]$, we have that $u(\lambda v_1 + (1-\lambda)v_2) \geq \lambda u(v_1) + (1-\lambda)u(v_2)$. Show that $S = \{v \in \mathbb{R}^n : \ u(v) \geq k\}$ is a convex set. For a concave $u : \mathbb{R}^2 \to \mathbb{R}$, the figure represents its graph $S = \{(x,y) \in \mathbb{R}^2 : \ u(x,y) \geq k\}$



(19) State the previous problem for a convex function $u : \mathbb{R}^n \to \mathbb{R}$.

(20) Determine the domains of the plane where the following functions are convex or concave.
(a) $f(x,y) = (x-1)^2 + xy^2$.
(b) $g(x,y) = \frac{x^3}{3} - 4xy + 12x + y^2$.
(c) $h(x,y) = e^{-x} + e^{-y}$.
(d) $k(x,y) = e^{xy}$.
(e) $l(x,y) = \ln\sqrt{xy}$.

(21) Determine the values of the parameters a and b so that the following functions are convex in their domains.
(a) $f(x,y,z) = ax^2 + y^2 + 2z^2 - 4axy + 2yz$
(b) $g(x,y) = 4ax^2 + 8xy + by^2$

(22) Discuss the concavity and convexity of the function $f(x,y) = -6x^2 + (2a+4)xy - y^2 + 4ay$ according to the values of a.

(23) Find the largest convex set of the plane where the function $f(x,y) = x^2 - y^2 - xy - x^3$ is concave.