January 23, 2019

MATHEMATICS FOR ECONOMICS II (2018-19)

ECONOMICS, LAW–ECONOMICS, INTERNATIONAL STUDIES–ECONOMICS

SHEET 1. MATRICES, DETERMINANTS AND SYSTEMS

(1) *Compute the following determinants:*

$$a)\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & 0 & 5 \end{vmatrix} \qquad b)\begin{vmatrix} 3 & -2 & 1 \\ 3 & 1 & 5 \\ 3 & 4 & 5 \end{vmatrix} \qquad c)\begin{vmatrix} 1 & 2 & 4 \\ 1 & -2 & 4 \\ 1 & 2 & -4 \end{vmatrix}$$

Solution: subtract solution to a) is -15, to part b) is -36 and to part c) is 32.

(2) *Use that* $\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} = 25$, *to compute the value of* $\begin{vmatrix} 2a & 2c & 2b \\ 2u & 2w & 2v \\ 2p & 2r & 2q \end{vmatrix}$.

Solution: In the determinant,

$$\begin{vmatrix} 2a & 2c & 2b \\ 2u & 2w & 2v \\ 2p & 2r & 2q \end{vmatrix}$$

we take out the common number 2 in each row,

$$2^3\begin{vmatrix} a & c & b \\ u & w & v \\ p & r & q \end{vmatrix}$$

Now swap columns 2 and 3

$$-2^3\begin{vmatrix} a & b & c \\ u & v & w \\ p & q & r \end{vmatrix}$$

Finally, swap rows 2 and 3,

$$2^3\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} = 2^3 \times 25 = 200$$

(3) *Verify the following identities without expanding the determinants:*

$$a)\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} \qquad b)\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$$

Solution: a): In the determinant

$$\left|\begin{pmatrix} bc & a & a^2 \\ ac & b & b^2 \\ ab & c & c^2 \end{pmatrix}\right|$$

we multiply and divide by a to obtain

$$\frac{1}{a}a\begin{vmatrix} bc & a & a^2 \\ ac & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \frac{1}{a}\begin{vmatrix} abc & a^2 & a^3 \\ ac & b & b^2 \\ ab & c & c^2 \end{vmatrix}$$

Now, we do the same procedure with b, and we observe that the determinant is the same as

$$\frac{1}{ab}\begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ ab & c & c^2 \end{vmatrix}$$

likewise with c,

$$\frac{1}{abc}\begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ abc & c^2 & c^3 \end{vmatrix}$$

Taking out abc in the first column, the last expression equals

$$\frac{1}{abc}abc\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

b): Adding the second column to the third,

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix}$$

we obtain

$$\begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} = 0$$

since columns 1 and 3 are the same.

(4) *Solve the following equation using the properties of the determinants:*

$$\begin{vmatrix} a & b & c \\ a & x & c \\ a & b & x \end{vmatrix} = 0$$

Solution: We must assume that $a \neq 0$. Otherwise, the determinant is 0 for every real number x. Since,

$$\begin{vmatrix} a & b & c \\ a & x & c \\ a & b & x \end{vmatrix} = a\begin{vmatrix} 1 & b & c \\ 1 & x & c \\ 1 & b & x \end{vmatrix}$$

the statement is equivalent (assuming $a \neq 0$) to

$$\begin{vmatrix} 1 & b & c \\ 1 & x & c \\ 1 & b & x \end{vmatrix} = 0$$

We subtract the first row to the second and third rows to obtain the following equation, ecuación

$$0 = \begin{vmatrix} 1 & b & c \\ 0 & x-b & 0 \\ 0 & 0 & x-c \end{vmatrix} = (x-b)(x-c)$$

Hence, the solutions are $x = b$ y $x = c$.

(5) *Simplify and compute the following expressions:*

$$a)\ \begin{vmatrix} ab & 2b^2 & -bc \\ a^2c & 3abc & 0 \\ 2ac & 5bc & 2c^2 \end{vmatrix} \qquad b)\ \begin{vmatrix} x & x & x & x \\ x & a & a & a \\ x & a & b & b \\ x & a & b & c \end{vmatrix} \qquad c)\ \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

Solution: a): In the determinant

$$\begin{vmatrix} ab & 2b^2 & -bc \\ a^2c & 3abc & 0 \\ 2ac & 5bc & 2c^2 \end{vmatrix}$$

we take out a in the first column, b in the second one and c in the third one,

$$abc\begin{vmatrix} b & 2b & -b \\ ac & 3ac & 0 \\ 2c & 5c & 2c \end{vmatrix}$$

and now we take out b in the first row, a in the second one and c in the third row,

$$a^2b^2c^2\begin{vmatrix} 1 & 2 & -1 \\ c & 3c & 0 \\ 2 & 5 & 2 \end{vmatrix}$$

Finally, we take out c in the second row

$$a^2b^2c^3\begin{vmatrix} 1 & 2 & -1 \\ 1 & 3 & 0 \\ 2 & 5 & 2 \end{vmatrix} = 3a^2b^2c^3$$

b): In the determinant

$$\left|\begin{pmatrix} x & x & x & x \\ x & a & a & a \\ x & a & b & b \\ x & a & b & c \end{pmatrix}\right|$$

we take out x in the first row

$$x\begin{vmatrix} 1 & 1 & 1 & 1 \\ x & a & a & a \\ x & a & b & b \\ x & a & b & c \end{vmatrix}$$

Now we subtract the first row times x, from the other rows,

$$x\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & a-x & a-x & a-x \\ 0 & a-x & b-x & b-x \\ 0 & a-x & b-x & c-x \end{vmatrix}$$

Expand now the determinant using the first column,

$$x\begin{vmatrix} a-x & a-x & a-x \\ a-x & b-x & b-x \\ a-x & b-x & c-x \end{vmatrix}$$

Subtract the first row from the other rows

$$x\left|\begin{pmatrix} a-x & a-x & a-x \\ 0 & b-a & b-a \\ 0 & b-a & c-a \end{pmatrix}\right|$$

we take out $a-x$ in the first row

$$x(a-x)\begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & b-a \\ 0 & b-a & c-a \end{vmatrix}$$

Expand now the determinant using the first column,

$$x(a-x)\begin{vmatrix} b-a & b-a \\ b-a & c-a \end{vmatrix} = x(a-x)(b-a)\begin{vmatrix} 1 & 1 \\ b-a & c-a \end{vmatrix} = x(a-x)(b-a)(c-b)$$

where we have taken out $b-a$ in the first row.

c): In the determinant

$$\left|\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}\right|$$

we subtract the last row from rows 2 and 3 and we expand it using the first column

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix}$$

we add now the first row to the second one and expand using the first column again,

$$-\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = -(1+2) = -3$$

(6) *Let A be a square matrix of order $n \times n$ such that $A^T A = I_n$. Show that $|A| = \pm 1$.*

Solution: Note that $1 = |I| = |A^t A| = |A^t||A| = |A|^2$. Hence, $|A|^2 = 1$ and the only possible values for the determinant are $|A| = 1$ o $|A| = -1$.

(7) *Find the rank of the following matrices:*

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 0 & 1 \\ -1 & 0 & 2 & 1 \end{pmatrix}$$

Solution: The rank

$$\operatorname{rg} A = \operatorname{rg}\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 3 \end{pmatrix}$$

is the same as

$$\operatorname{rg}\begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & -3 & -1 \end{pmatrix} = \operatorname{rg}\begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \end{pmatrix} = 2.$$

On the other hand,

$$\operatorname{rg}\begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 0 & 1 \\ -1 & 0 & 2 & 1 \end{pmatrix} = \operatorname{rg}\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -4 & -1 \\ 0 & 1 & 4 & 2 \end{pmatrix} = \operatorname{rg}\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -4 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 3$$

(8) *Study the rank of the following matrices, depending on the possible values of x:*

$$A = \begin{pmatrix} x & 0 & x^2 & 1 \\ 1 & x^2 & x^3 & x \\ 0 & 0 & 1 & 0 \\ 0 & 1 & x & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & x & x^2 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \qquad C = \begin{pmatrix} x & -1 & 0 & 1 \\ 0 & x & -1 & 1 \\ 1 & 0 & -1 & 2 \end{pmatrix}$$

Solution: We compute first the rank of A.

$$\left|\begin{pmatrix} 1 & x^2 & x^3 \\ 0 & 0 & 1 \\ 0 & 1 & x \end{pmatrix}\right| = \begin{vmatrix} 0 & 1 \\ 1 & x \end{vmatrix} = -1$$

so $\operatorname{rg} A \geq 3$ for any value of x. Expanding the determinant of A using the third row

$$|A| = \begin{vmatrix} x & 0 & 1 \\ 1 & x^2 & x \\ 0 & 1 & 0 \end{vmatrix}$$

Now we expand the determinant using row 3

$$|A| = -\begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} = -(x^2-1)$$

so the rank of A is 4 if $x^2 \neq 1$, that is if $x \neq 1$ and $x \neq -1$. To sum up,

$$\operatorname{rg} A = \begin{cases} 3, & \text{if } x = 1 \text{ o } x = -1; \\ 4, & \text{in all other cases.} \end{cases}$$

Now we compute the rank of B. Note that the minor

$$\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1$$

does not vanish. Hence, $\operatorname{rg} B \geq 2$. On the other hand,

$$\operatorname{rg} B = \operatorname{rg}\left(\begin{pmatrix} 1 & x & x^2 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}\right) = \operatorname{rg}\begin{pmatrix} 1 & x & x^2 \\ 1 & 2 & 4 \\ 0 & 1 & 5 \end{pmatrix} = \operatorname{rg}\begin{pmatrix} 1 & x & x^2 \\ 0 & 2-x & 4-x^2 \\ 0 & 1 & 5 \end{pmatrix}$$

and this rank is 3, unless $5(2-x) = 4-x^2$. This happens if and only if $x^2 - 5x + 6 = 0$, that is if

$$x = \frac{5 \pm \sqrt{25-24}}{2} = 2, 3$$

To sum up,

$$\operatorname{rg} B = \begin{cases} 2, & \text{si } x = 2 \text{ o } x = 3; \\ 3, & \text{en los demás casos.} \end{cases}$$

Finally, we compute the rank of C.

$$\operatorname{rg} C = \operatorname{rg}\begin{pmatrix} x & -1 & 0 & 1 \\ 0 & x & -1 & 1 \\ 1 & 0 & -1 & 2 \end{pmatrix} = \operatorname{rg}\begin{pmatrix} 1 & 0 & -1 & 2 \\ x & -1 & 0 & 1 \\ 0 & x & -1 & 1 \end{pmatrix} =$$

$$\operatorname{rg}\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & x & 1-2x \\ 0 & x & -1 & 1 \end{pmatrix} = \operatorname{rg}\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & x & 1-2x \\ 0 & 0 & x^2-1 & 1+x-2x^2 \end{pmatrix}$$

ant this rank es 3 unless $x^2 - 1 = 0$ and $1 + x - 2x^2 = 0$. The solutions to $x^2 - 1 = 0$ are $x = 1$ and $x = -1$. The solutions to $2x^2 - x - 1$ are

$$x = \frac{1 \pm \sqrt{1+8}}{4} = 1, -\frac{1}{2}$$

Hence, $x = 1$ is the only solution to both equations. Therefore,

$$\operatorname{rg} C = \begin{cases} 2, & \text{if } x = 1; \\ 3, & \text{otherwise.} \end{cases}$$

(9) *Let A and B be square, invertible matrices of the same order. Solve for X in the following matrix equations:*
(a) $X^t \cdot A = B$
(b) $(X \cdot A)^{-1} = A^{-1} \cdot B$

Find X in the preceding equations when $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 3 & 1 & 2 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Solution: a): Taking transpose in the equation $X^tA = B$, we obtain $B^t = (X^tA)^t = A^tX$. Solving for X we have that $X = (A^t)^{-1}B^t = (A^{-1})^tB^t$. When

$$A = \left(\left(\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 3 & 1 & 2 \end{pmatrix}\right)\right) \qquad B = \left(\left(\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}\right)\right)$$

we see that

$$X = (A^{-1})^tB^t = \begin{pmatrix} 6 & 1 & 1 \\ -9 & -2 & -1 \\ -2 & 0 & 0 \end{pmatrix}$$

b): Taking inverse matrices in the equation $(XA)^{-1} = A^{-1}B$ we see that $XA = (A^{-1}B)^{-1} = B^{-1}(A^{-1})^{-1} = B^{-1}A$. And since, A is invertible, we can solve for $X = B^{-1}AA^{-1} = B^{-1}$. When

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 3 & 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

we obtain

$$X = B^{-1} = \begin{pmatrix} -1/2 & 1/2 & 1/2 \\ 0 & -1 & 1 \\ 1/2 & 1/2 & -1/2 \end{pmatrix}$$

(10) *Whenever possible, compute the inverse of the following matrices:*

$$A = \begin{pmatrix} 1 & 0 & x \\ -x & 1 & -\frac{x^2}{2} \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & x & 3 \\ 4 & 1 & -x \end{pmatrix}$$

Solution: We use the formula to compute the inverse of de A que

$$A^{-1} = \begin{pmatrix} 1 & 0 & -x \\ x & 1 & \frac{-x^2}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

We may also compute the inverse by performing elementary operations by rows,

$$\left(\begin{array}{ccc|ccc} 1 & 0 & x & 1 & 0 & 0 \\ -x & 1 & -\frac{x^2}{2} & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Bow we add to the second row the first one times x,

$$\left(\begin{array}{ccc|ccc} 1 & 0 & x & 1 & 0 & 0 \\ 0 & 1 & \frac{x^2}{2} & x & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

subtract the third row times x to the first one,

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -x \\ 0 & 1 & \frac{x^2}{2} & x & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Substract now the third row times $\frac{x^2}{2}$ to the second row,

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -x \\ 0 & 1 & 0 & x & 1 & -\frac{x^2}{2} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

from here we obtain the inverse matrix o A.

Now we compute the inverse matrix of

$$B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & x & 3 \\ 4 & 1 & -x \end{pmatrix}$$

By the usual formula we see that

$$B^{-1} = \frac{1}{x^2 - 4x + 3} \begin{pmatrix} x^2+3 & 1 & -x \\ -12 & x-4 & 3 \\ 4x & 1 & -x \end{pmatrix}$$

We also compute the inverse matrix by noticing that

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & t & 3 & 0 & 1 & 0 \\ 4 & 1 & -t & 0 & 0 & 1 \end{array}\right)$$

now we subtract from the third row the first row times 4,

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & t & 3 & 0 & 1 & 0 \\ 0 & 1 & 4-t & -4 & 0 & 1 \end{array}\right)$$

exchange rows 2 and 3

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4-t & -4 & 0 & 1 \\ 0 & t & 3 & 0 & 1 & 0 \end{array}\right)$$

Add the second row times t to the third row

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4-t & -4 & 0 & 1 \\ 0 & 0 & t^2-4t+3 & 4t & 1 & -t \end{array}\right)$$

From here we see that $|A| = t^2 - 4t + 3$. The roots of this polynomial are

$$t = \frac{4 \pm \sqrt{16-12}}{2} = 1, 3$$

so the inverse matrix exists if and only if $t \neq 1$ y $t \neq 3$. Assuming this inequality, divide by $t^2 - 4t + 3$ the last row

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4-t & -4 & 0 & 1 \\ 0 & 0 & 1 & \frac{4t}{t^2-4t+3} & \frac{1}{t^2-4t+3} & -\frac{t}{t^2-4t+3} \end{array}\right)$$

and add the third row to the first one and third row times $t-4$ to the second row,

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{t^2+3}{t^2-4t+3} & \frac{1}{t^2-4t+3} & \frac{-t}{t^2-4t+3} \\ 0 & 1 & 0 & \frac{-12}{t^2-4t+3} & \frac{t-4}{t^2-4t+3} & \frac{3}{t^2-4t+3} \\ 0 & 0 & 1 & \frac{4t}{t^2-4t+3} & \frac{1}{t^2-4t+3} & \frac{-t}{t^2-4t+3} \end{array}\right)$$

so

$$A^{-1} = \frac{1}{t^2 - 4t + 3} \begin{pmatrix} t^2+3 & 1 & -t \\ -12 & t-4 & 3 \\ 4t & 1 & -t \end{pmatrix}$$

(11) *Whenever possible, compute the inverse of the following matrices:*

$$A = \begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix} \qquad C = \begin{pmatrix} 4 & 5 & -2 \\ -2 & -2 & 1 \\ -1 & -1 & 1 \end{pmatrix}$$

Solution: We shall use Gauss' method to compute the inverse. We begin with the matrix

$$(A|I) = \left(\begin{array}{rrr|rrr} 4 & 6 & 0 & 1 & 0 & 0 \\ -3 & -5 & 0 & 0 & 1 & 0 \\ -3 & -6 & 1 & 0 & 0 & 1 \end{array}\right)$$

divide the first row by 4

$$\left(\begin{array}{rrr|rrr} 1 & 6/4 & 0 & 1/4 & 0 & 0 \\ -3 & -5 & 0 & 0 & 1 & 0 \\ -3 & -6 & 1 & 0 & 0 & 1 \end{array}\right)$$

now, subtract the third row to the second one

$$\left(\begin{array}{rrr|rrr} 1 & 6/4 & 0 & 1/4 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ -3 & -6 & 1 & 0 & 0 & 1 \end{array}\right)$$

add the third row to the first one times 3,

$$\left(\begin{array}{lrr|rrr} 1 & 3/2 & 0 & 1/4 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & -3/2 & 1 & 3/4 & 0 & 1 \end{array}\right)$$

add the third row to the first one

$$\left(\begin{array}{lrr|rrr} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & -3/2 & 1 & 3/4 & 0 & 1 \end{array}\right)$$

add the second row times 3/2 to the third row,

$$\left(\begin{array}{rrr|rrr} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & -1/2 & 3/4 & 3/2 & -1/2 \end{array}\right)$$

multiply the third row by -2

$$\left(\begin{array}{rrr|rrr} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -3/2 & -3 & 1 \end{array}\right)$$

add the second row to the third one and subtract them to the first one

$$\left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 5/2 & 3 & 0 \\ 0 & 1 & 0 & -3/2 & -2 & 0 \\ 0 & 0 & 1 & -3/2 & -3 & 1 \end{array}\right)$$

Thus, the inverse is

$$\begin{pmatrix} 5/2 & 3 & 0 \\ -3/2 & -2 & 0 \\ -3/2 & -3 & 1 \end{pmatrix}$$

Note that $|B| = 0$, so B does not have an inverse.

We compute the inverse matrix of C using Gauss' methor

$$(C|\,I) = \left(\begin{array}{rrr|rrr} 4 & 5 & -2 & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

Multiply the third row by -1 and exchange rows 1 and 3

$$(C|\,I) = \left(\begin{array}{rrr|rrr} 1 & 1 & -1 & 0 & 0 & -1 \\ -2 & -2 & 1 & 0 & 1 & 0 \\ 4 & 5 & -2 & 1 & 0 & 0 \end{array}\right)$$

add the second row times 2 to the third row

$$(C|\,I) = \left(\begin{array}{rrr|rrr} 1 & 1 & -1 & 0 & 0 & -1 \\ -2 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 & 0 \end{array}\right)$$

add the first row times 2 to the first one

$$(C|\,I) = \left(\begin{array}{rrr|rrr} 1 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 & 2 & 0 \end{array}\right)$$

exchange rows 2 and 3

$$(C|\,I) = \left(\begin{array}{rrr|rrr} 1 & 1 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 1 & -2 \end{array}\right)$$

The third row times -1,

$$(C|\,I) = \left(\begin{array}{rrr|rrr} 1 & 1 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 \end{array}\right)$$

add the row 3 to row 1,

$$(C|\,I) = \left(\begin{array}{rrr|rrr} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 \end{array}\right)$$

subtract row 2 to row 1

$$(C|\,I) = \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -1 & -3 & 1 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 \end{array}\right)$$

the inverse is

$$C^{-1} = \begin{pmatrix} -1 & -3 & 1 \\ 1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}$$

(12) *Given the system* $\left\{\begin{array}{l} mx - y = 1 \\ x - my = 2m - 1 \end{array}\right.$ *, find* m *so that the system*

(a) *has no solution,*
(b) *has infinitely many solutions,*
(c) *has a unique solution,*
(d) *has a solution with* $x = 3$.

Solution: The matrix associated to the system is

$$\left(\begin{array}{cc|c} m & -1 & 1 \\ 1 & -m & 2m-1 \end{array}\right)$$

whose rank is the same as

$$\operatorname{rg}\left(\begin{array}{cc|c} 1 & -m & 2m-1 \\ m & -1 & 1 \end{array}\right) = \operatorname{rg}\left(\begin{array}{cc|c} 1 & -m & 2m-1 \\ 0 & m^2-1 & 1+m-2m^2 \end{array}\right)$$

Therefore, the rank of this matrix 2 is $m^2 \neq 1$. That is, if $m \neq 1$ and $m \neq -1$ then the system has a unique solution. In this case the system is equivalent to the following one

$$\begin{aligned} x - my &= 2m-1 \\ (m^2-1)y &= 1+m-2m^2 \end{aligned}$$

and the solution is

$$\begin{aligned} y &= \frac{1+m-2m^2}{m^2-1} = \frac{-(m-1)(1+2m)}{(m-1)(m+1)} = \frac{-1-2m}{m+1} \\ x &= 2m-1+my = 2m-1-m\frac{1+2m}{m+1} = \frac{-1}{m+1} \end{aligned}$$

If $x = 3$ is a solution we must have that

$$3 = \frac{-1}{m+1}$$

this implies that $m = -4/3$.

We study now the case $m = 1$.

$$\operatorname{rg}\left(\begin{array}{cc|c} 1 & -m & 2m-1 \\ 0 & m^2-1 & 1+m-2m^2 \end{array}\right) = \operatorname{rg}\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right) = 1 = \operatorname{rg} A$$

and the system has a unique solution. The original system of equations is equivalent to the following one

$$x - y = 1$$

and the set of solutions is $\{1+y, y) : y \in R\}$. Taking $y = 2$, we obtain the solution $(3, 2)$.

For the case $m = -1$ we have that

$$\operatorname{rg}\left(\begin{array}{cc|c} 1 & -m & 2m-1 \\ 0 & m^2-1 & 1+m-2m^2 \end{array}\right) = \operatorname{rg}\left(\begin{array}{cc|c} 1 & 1 & -3 \\ 0 & 0 & -2 \end{array}\right) = 2 \neq \operatorname{rg} A$$

and the system is inconsistent.

(13) *Given the system of linear equations* $\left\{\begin{array}{l} x + ay = 1 \\ ax + z = 1 \\ ay + z = 2 \end{array}\right.$

(a) *Express it in matrix form;*

(b) *Write the unknowns vector, the independent term and the associated homogeneous system;*

Solution: En forma matricial, el sistema queda expresado como $AX = B$ con

$$A = \begin{pmatrix} 1 & a & 0 \\ a & 0 & 1 \\ 0 & a & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(c) *Discuss and solve it according to the values of a.*
Solution: The augmented matrix is

$$(A|B) = \left(\begin{array}{ccc|c} 1 & a & 0 & 1 \\ a & 0 & 1 & 1 \\ 0 & a & 1 & 2 \end{array}\right)$$

First, we swap rows 2 and 3,

$$\operatorname{rg}(A|B) = \operatorname{rg}\left(\begin{array}{ccc|c} 1 & a & 0 & 1 \\ 0 & a & 1 & 2 \\ a & 0 & 1 & 1 \end{array}\right)$$

Now, we subtract row 1 times a from row 3

$$\operatorname{rg}(A|B) = \operatorname{rg}\left(\begin{array}{ccc|c} 1 & a & 0 & 1 \\ 0 & a & 1 & 2 \\ 0 & -a^2 & 1 & 1-a \end{array}\right)$$

We add row 2 times a to row 2,

$$\operatorname{rg}(A|B) = \operatorname{rg}\left(\begin{array}{ccc|c} 1 & a & 0 & 1 \\ 0 & a & 1 & 2 \\ 0 & 0 & 1+a & 1+a \end{array}\right)$$

From this, we see that if $a \neq 0$ y $a \neq -1$ then $\operatorname{rg}(A) = \operatorname{rg}(A|B) = 3 =$ the number of unknowns, so the system has a unique solution. In this case, the original system is equivalent to the following one,

$$\begin{aligned} x + ay &= 1 \\ ay + z &= 2 \\ (1+a)z &= 1+a \end{aligned}$$

and we obtain the solution $z = 1$, $y = 1/a$, $x = 0$.
If $a = -1$, then

$$\operatorname{rg}(A|B) = \operatorname{rg}\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

from this we see that $\operatorname{rg}(A) = \operatorname{rg}(A|B) = 2$ which is strictly less than the number of unknowns, so the system is undetermined. Now, the original system is equivalent to the following one,

$$\begin{aligned} x - y &= 1 \\ -y + z &= 2 \end{aligned}$$

We take z as the parameter and solve for the variables

$$y = z - 2 \qquad x = 1 + y = z - 1$$

subtract set of solutions is

$$\{(z-1, z-2, z) \in R^3 : z \in R\}$$

Finally, if $a = 0$, then

$$\operatorname{rg}(A|B) = \operatorname{rg}\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

and we see that $\operatorname{rg}(A) = 2 < \operatorname{rg}(A|B) = 3$, so the system is inconsistent.

(14) *Discuss and solve the system* $\left\{\begin{array}{r} x+y+z+2t-w=1 \\ -x-2y+2w=-2 \\ x+2z+4t=0 \end{array}\right.$

Solution: The augmented matrix $A|B)$ is

$$(A|B)=\left(\begin{array}{rrrrr|r} 1 & 1 & 1 & 2 & -1 & 1 \\ -1 & -2 & 0 & 0 & 2 & -2 \\ 1 & 0 & 2 & 4 & 0 & 0 \end{array}\right)$$

By performing elementary row operations we see that

$$\operatorname{rg}(A|B)=\operatorname{rg}\left(\begin{array}{rrrrr|r} 1 & 1 & 1 & 2 & -1 & 1 \\ 0 & -1 & 1 & 2 & 1 & -1 \\ 0 & -1 & 1 & 2 & 1 & -1 \end{array}\right)=\operatorname{rg}\left(\begin{array}{rrrrr|r} 1 & 1 & 1 & 2 & -1 & 1 \\ 0 & -1 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

so $\operatorname{rg}(A)=\operatorname{rg}(A|B)=2$ which is less than the number of unknowns. The system is consistent and underdetermined. Since there are five unknowns, we obtain $5-2=3$ parameters. We have to solve the following linear system,

$$\begin{array}{rcl} x+y+z+2t-w & = & 1 \\ -y+z+2t+w & = & -1 \end{array}$$

We choose z, w and t as a parameters and solve for $y=z+2t+w+1$, $x=1-y-z-2t-w=-2z-4t$. The set of solutions is

$$\{(-2z-4t, z+2t+w+1, z, t, w)\in\mathbb{R}^5 : z,t,w\in\mathbb{R}\}$$

(15) *Discuss and solve the following system, according the values of a and b:*

$$\left\{\begin{array}{r} x+y+z=0 \\ ax+y+z=b \\ 2x+2y+(a+1)\,z=0 \end{array}\right.$$

Solution: The augmented matrix is

$$\operatorname{rg}(A|B)=\operatorname{rg}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ a & 1 & 1 & b \\ 2 & 2 & a+1 & 0 \end{array}\right)$$

Performing elementary row operations,

$$\operatorname{rg}(A|B)=\operatorname{rg}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1-a & 1-a & b \\ 0 & 0 & a-1 & 0 \end{array}\right)$$

If $a\neq 1$, then the system has a unique solution and is equivalent to the following one

$$\begin{array}{rcl} x+y+z & = & 0 \\ y+z & = & \dfrac{b}{1-a} \\ (a-1)z & = & 0 \end{array}$$

The solution is

$$z=0, \quad y=\frac{b}{1-a}, \quad x=\frac{-b}{1-a}$$

Now, we study the linear system for the value $a = 1$. In this case,

$$\operatorname{rg}(A|B) = \operatorname{rg}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So, when $b \neq 0$ the system is inconsistent, since $\operatorname{rg}(A) = 1$, $\operatorname{rg}(A|B) = 2$.

Finally, if $a = 1$ and $b = 0$ then $\operatorname{rg}(A) = \operatorname{rg}(A|B) = 1$ and the system is consistent and underdetermined with $3 - 1 = 2$ parameters. The original system is equivalent to the following one

$$x + y + z = 0$$

Taking y and z as parameters, the set of solutions is

$$\{(-y - z, y, z) : y, z \in R\}$$

To sum up,

$$\left\{\begin{array}{ll} a \neq 1, & \text{There is a unique solution: } z = 0, \quad y = \frac{b}{1-a}, \quad x = \frac{-b}{1-a}; \\ a = 1, & \text{If} \left\{\begin{array}{ll} b \neq 0, & \text{inconsistent;} \\ b = 0, & \text{undetermined. The set of solutions is the set } \{(-y-z, y, z) : y, z \in R\}. \end{array}\right. \end{array}\right. .$$

(16) *Discuss and solve the following system, according the values of a and b:*

$$\left\{\begin{array}{r} x - 2y + bz = 3 \\ 5x + 2y = 1 \\ ax + z = 2 \end{array}\right.$$

Solution: The augmented matrix is

$$(A|B) = \left(\begin{array}{rrr|r} 1 & -2 & b & 3 \\ 5 & 2 & 0 & 1 \\ a & 0 & 1 & 2 \end{array}\right)$$

Performing elementary row operations we see that

$$\operatorname{rg}(A|B) = \operatorname{rg}\left(\begin{array}{ccc|c} 1 & -2 & b & 3 \\ 0 & 12 & -5b & -14 \\ 0 & 2a & 1 - ab & 2 - 3a \end{array}\right)$$

We subtract row 2 times $a/6$ from row 3,

$$\operatorname{rg}(A|B) = \operatorname{rg}\left(\begin{array}{ccc|c} 1 & -2 & b & 3 \\ 0 & 12 & -5b & -14 \\ 0 & 0 & 1 - \frac{ab}{6} & 2 - \frac{2a}{3} \end{array}\right)$$

And we see that if $ab \neq 6$ then the unique solution is

$$z = \frac{12 - 4a}{6 - ab}, \quad y = \frac{-7}{6} + \frac{5}{12}\frac{12 - 4a}{6 - ab}b, \quad x = 3 + \frac{12 - 4a}{6 - ab}b - \frac{7}{3} + \frac{5}{6}\frac{12 - 4a}{6 - ab}b$$

If $ab = 6$ we solve for $a = 6/b$ (since $b \neq 0$) to obtain

$$\operatorname{rg}(A|B) = \operatorname{rg}\left(\begin{array}{ccc|c} 1 & -2 & b & 3 \\ 0 & 12 & -5b & -14 \\ 0 & 0 & 0 & \frac{2b-4}{b} \end{array}\right)$$

So, if $b = 2$ (and hence $a = 3$) then $\operatorname{rg}(A|B) = \operatorname{rg}(A) = 2$ and the system is consistent and underdetermined. It is equivalent to the following system

$$\begin{aligned} x - 2y + 2z &= 3 \\ 12y - 10z &= -14 \end{aligned}$$

Taking z as the parameter, the set of solutions is

$$\{(\frac{2-z}{3}, \frac{5z-7}{6}, z) : z \in R\}$$

Finally, if $b \neq 2$ then the system is inconsistent.

(17) *Using Cramer's method, solve the following system:*

$$\begin{cases} -x+y+z=3 \\ x-y+z=7 \\ x+y-z=1 \end{cases}$$

Solution: The determinant of the associated matrix is

$$\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4$$

And the solutions are

$$x = \frac{1}{4}\begin{vmatrix} 3 & 1 & 1 \\ 7 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \frac{16}{4} = 4 \quad y = \frac{1}{4}\begin{vmatrix} -1 & 3 & 1 \\ 1 & 7 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \frac{8}{4} = 2 \quad z = \frac{1}{4}\begin{vmatrix} -1 & 1 & 3 \\ 1 & -1 & 7 \\ 1 & 1 & 1 \end{vmatrix} = \frac{20}{4} = 5$$

(18) *Using Cramer's method, solve the following system*

$$\begin{cases} x+y=12 \\ y+z=8 \\ x+z=6 \end{cases}$$

Solution: The determinant of the associated matrix is

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2$$

And the solutions are

$$x = \frac{1}{2}\begin{vmatrix} 12 & 1 & 0 \\ 8 & 1 & 1 \\ 6 & 0 & 1 \end{vmatrix} = \frac{10}{2} = 5 \quad y = \frac{1}{2}\begin{vmatrix} 1 & 12 & 0 \\ 0 & 8 & 1 \\ 1 & 6 & 1 \end{vmatrix} = \frac{14}{2} = 7 \quad z = \frac{1}{2}\begin{vmatrix} 1 & 1 & 12 \\ 0 & 1 & 8 \\ 1 & 0 & 6 \end{vmatrix} = \frac{2}{2} = 1$$

(19) *Using Cramers's method, solve the following system:*

$$\begin{cases} x+y-2z=9 \\ 2x-y+4z=4 \\ 2x-y+6z=-1 \end{cases}$$

Solution: The determinant of the associated matrix is

$$\begin{vmatrix} 1 & 1 & -2 \\ 2 & -1 & 4 \\ 2 & -1 & 6 \end{vmatrix} = -6$$

And the solutions are

$$x = \frac{1}{-6}\begin{vmatrix} 9 & 1 & -2 \\ 4 & -1 & 4 \\ 1 & -1 & 6 \end{vmatrix} = \frac{-36}{-6} = 6 \quad y = \frac{1}{-6}\begin{vmatrix} 1 & 9 & -2 \\ 2 & 4 & 4 \\ 2 & 1 & 6 \end{vmatrix} = \frac{12}{-6} = -2 \quad z = \frac{1}{-6}\begin{vmatrix} 1 & 1 & 9 \\ 2 & -1 & 4 \\ 2 & -1 & 1 \end{vmatrix} = \frac{15}{-6} = \frac{-5}{2}$$

(20) *Given the system of equations*

$$\begin{cases} x + 2y + z = 3 \\ ax + (a+3)y + 3z = 1 \end{cases}$$

(a) *Study for what values of a the system is not consistent.*
(b) *For each value of a for which the system is consistent, find the solution.*

Solution: The rank of the associated matrix is

$$\operatorname{rg}(A|B) = \operatorname{rg}\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ a & a+3 & 3 & 1 \end{array}\right) = \operatorname{rg}\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 3-a & 3-a & 1-3a \end{array}\right)$$

We see that if $a = 3$ then the system has no solutions because $\operatorname{rg}(A|B) = 2$, $\operatorname{rg}(A) = 1$. If $a \neq 3$ the system is consistent and underdetermined. It is equivalent to the following one

$$\begin{aligned} x + 2y + z &= 3 \\ (3-a)y + (3-a)z &= 1 - 3a \end{aligned}$$

Taking z as the parameter, the set of solutions is

$$\{(\frac{7+3a}{3-a} + z, \frac{1-3a}{3-a} - z, z) : z \in R\}$$

(21) *Given the homogeneous system*

$$\begin{cases} 3x + 3y - z = 0 \\ -4x - 2y + mz = 0 \\ 3x + 4y + 6z = 0 \end{cases}$$

(a) *Find m so that it admits solutions other than the trivial solution.*
(b) *Solve the system.*

Solution: The rank of the associated matrix is

$$\operatorname{rg}\begin{pmatrix} 3 & 3 & -1 \\ -4 & -2 & m \\ 3 & 4 & 6 \end{pmatrix} = \operatorname{rg}\begin{pmatrix} 3 & 3 & -1 \\ -1 & 1 & m-1 \\ 0 & 1 & 7 \end{pmatrix}$$

we swap the first two rows

$$= \operatorname{rg}\begin{pmatrix} -1 & 1 & m-1 \\ 3 & 3 & -1 \\ 0 & 1 & 7 \end{pmatrix} = \operatorname{rg}\begin{pmatrix} -1 & 1 & m-1 \\ 0 & 6 & 3m-4 \\ 0 & 1 & 7 \end{pmatrix}$$

and now swap the last two rows

$$= \operatorname{rg}\begin{pmatrix} -1 & 1 & m-1 \\ 0 & 1 & 7 \\ 0 & 6 & 3m-4 \end{pmatrix} = \operatorname{rg}\begin{pmatrix} -1 & 1 & m-1 \\ 0 & 1 & 7 \\ 0 & 0 & 3m-46 \end{pmatrix}$$

The system has non-trivial solution if and only if the determinant vanishes. This happens for the value $m = 46/3$. In this case, the original system is equivalent to the following one

$$\begin{aligned} -x + y + \frac{43}{3}z &= 0 \\ y + 7z &= 0 \end{aligned}$$

and the set of solutions is

$$\{(22z/3, -7z, z) : z \in \mathbb{R}\}$$