January 23, 2019

MATHEMATICS FOR ECONOMICS II (2018-19)

ECONOMICS, LAW–ECONOMICS, INTERNATIONAL STUDIES–ECONOMICS

SHEET 1. MATRICES, DETERMINANTS AND SYSTEMS

(1) Compute the following determinants:

$$\text{a)} \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & 0 & 5 \end{vmatrix} \qquad \text{b)} \begin{vmatrix} 3 & -2 & 1 \\ 3 & 1 & 5 \\ 3 & 4 & 5 \end{vmatrix} \qquad \text{c)} \begin{vmatrix} 1 & 2 & 4 \\ 1 & -2 & 4 \\ 1 & 2 & -4 \end{vmatrix}$$

(2) Use that $\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} = 25$, to compute the value of $\begin{vmatrix} 2a & 2c & 2b \\ 2u & 2w & 2v \\ 2p & 2r & 2q \end{vmatrix}$.

(3) Verify the following identities without expanding the determinants:

$$\text{a)} \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} \qquad \text{b)} \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$$

(4) Solve the following equation using the properties of the determinants:

$$\begin{vmatrix} a & b & c \\ a & x & c \\ a & b & x \end{vmatrix} = 0$$

(5) Simplify and compute the following expressions:

$$\text{a)} \begin{vmatrix} ab & 2b^2 & -bc \\ a^2c & 3abc & 0 \\ 2ac & 5bc & 2c^2 \end{vmatrix} \qquad \text{b)} \begin{vmatrix} x & x & x & x \\ x & a & a & a \\ x & a & b & b \\ x & a & b & c \end{vmatrix} \qquad \text{c)} \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

(6) Let A be a square matrix of order $n \times n$ such that $A^T A = I_n$. Show that $|A| = \pm 1$.

(7) Find the rank of the following matrices:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 0 & 1 \\ -1 & 0 & 2 & 1 \end{pmatrix}$$

(8) Study the rank of the following matrices, depending on the possible values of x:

$$A = \begin{pmatrix} x & 0 & x^2 & 1 \\ 1 & x^2 & x^3 & x \\ 0 & 0 & 1 & 0 \\ 0 & 1 & x & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & x & x^2 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \qquad C = \begin{pmatrix} x & -1 & 0 & 1 \\ 0 & x & -1 & 1 \\ 1 & 0 & -1 & 2 \end{pmatrix}$$

(9) Let A and B be square, invertible matrices of the same order. Solve for X in the following matrix equations:

(a) $X^t \cdot A = B$

(b) $(X \cdot A)^{-1} = A^{-1} \cdot B$

Find X in the preceding equations when $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 3 & 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

(10) Whenever possible, compute the inverse of the following matrices:

$$A = \begin{pmatrix} 1 & 0 & x \\ -x & 1 & -\frac{x^2}{2} \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & x & 3 \\ 4 & 1 & -x \end{pmatrix}$$

(11) Whenever possible, compute the inverse of the following matrices:

$$A = \begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix} \qquad C = \begin{pmatrix} 4 & 5 & -2 \\ -2 & -2 & 1 \\ -1 & -1 & 1 \end{pmatrix}$$

(12) Given the system $\left\{ \begin{array}{l} mx - y = 1 \\ x - my = 2m - 1 \end{array} \right.$, find m so that the system

(a) has no solution,
(b) has infinitely many solutions,
(c) has a unique solution,
(d) has a solution with $x = 3$.

(13) Given the system of linear equations $\left\{ \begin{array}{l} x + ay = 1 \\ ax + z = 1 \\ ay + z = 2 \end{array} \right.$

(a) Express it in matrix form;
(b) Write the unknowns vector, the independent term and the associated homogeneous system;
(c) Discuss and solve it according to the values of a.

(14) Discuss and solve the system $\left\{ \begin{array}{r} x + y + z + 2t - w = 1 \\ -x - 2y + 2w = -2 \\ x + 2z + 4t = 0 \end{array} \right.$

(15) Discuss and solve the following system, according the values of a and b:

$$\left\{ \begin{array}{r} x + y + z = 0 \\ ax + y + z = b \\ 2x + 2y + (a+1)\, z = 0 \end{array} \right.$$

(16) Discuss and solve the following system, according the values of a and b:

$$\left\{ \begin{array}{r} x - 2y + bz = 3 \\ 5x + 2y = 1 \\ ax + z = 2 \end{array} \right.$$

(17) Using Cramer's method, solve the following system:

$$\left\{ \begin{array}{r} -x + y + z = 3 \\ x - y + z = 7 \\ x + y - z = 1 \end{array} \right.$$

(18) Using Cramer's method, solve the following system

$$\left\{ \begin{array}{r} x + y = 12 \\ y + z = 8 \\ x + z = 6 \end{array} \right.$$

(19) Using Cramers's method, solve the following system:

$$\left\{ \begin{array}{r} x + y - 2z = 9 \\ 2x - y + 4z = 4 \\ 2x - y + 6z = -1 \end{array} \right.$$

(20) Given the system of equations

$$\left\{\begin{aligned} x + 2y + z &= 3 \\ ax + (a+3)y + 3z &= 1 \end{aligned}\right.$$

(a) Study for what values of a the system is not consistent.
(b) For each value of a for which the system is consistent, find the solution.

(21) Given the homogeneous system

$$\left\{\begin{aligned} 3x + 3y - z &= 0 \\ -4x - 2y + mz &= 0 \\ 3x + 4y + 6z &= 0 \end{aligned}\right.$$

(a) Find m so that it admits solutions other than the trivial solution.
(b) Solve the system.