SOLVED

$\boxed{1}$

Consider the following system of linear equations which depend on a and b

$$\begin{cases} x + 2y - 4z + (2b-2)t = 2 \\ \quad - y + bz - 2t = 2 \\ -x + (b-2)y - 5z + 2t = a \end{cases}$$

(a) (15 points) Classify the system in terms of the value of a and b. When the system is consistent (i.e. admits solutions), justify which is the number of parameters needed to describe the solutions.

(b) (5 points) Find the solution or solutions of the system when $b = -3$ and $a = 4$.

Solution:

(a) The expanded matrix A^* can be given the equivalent echelon form

$$\left(\begin{array}{cccc|c} 1 & 2 & -4 & (2b-2) & 2 \\ 0 & -1 & b & -2 & 2 \\ 0 & 0 & b^2-9 & 0 & a+2+2b \end{array}\right)$$

(Obtained with the following elementary operations on the matriz A^*: add the first row to the third one, and in the resulting matrix, to add to the third row the first one times b).

The rank of A (matrix of the system) and the rank of A^* is 3 if $b^2 \neq 9$, and it is 2 if $b^2 = 9$ and $a+2+2b = 0$. Otherwise, the matrices A and A^* have unequal ranks.

In summary, the system admits infinitely many solutions when

- $b^2 \neq 9$, with solutions depending on one parameter
- $b^2 = 9$ y $a + 2 + 2b = 0$ (that is, $b = 3$ and $a = -8$, or $b = -3$ and $a = 4$), with solutions depending on two parameters

The system has no solutions when $b^2 = 9$ and $a + 2 + 2b \neq 0$

(b) Choosing parameters z and t, the solutions are $(x, y, z, t) = (6 + 10z + 12t, -2 - 3z - 2t, z, t)$.

$\boxed{2}$

Consider the matrix

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

(a) (10 points) Calculate eigenvalues and eigenvectors of A. Is the matriz diagonalizable?

(b) (10 points) Matrix A can be considered the matrix of a quadratic form Q. Classify Q in the following two cases:

(i) In $\mathbb{R}^3$;

(ii) restricted to the plane $2x + y = 0$.

Solution:

(a) The eigenvalues are the roots of the characteristic polynomial $-(2+\lambda)\,(-(2-\lambda)(1+\lambda)-4)$, obtaining $\lambda = -2$ double and $\lambda = 3$, simple.

A is diagonalizable since it is symmetric (or because we can construct an invertible matrix P such that $P^{-1}AP$ is diagonal, see below).

The proper subspaces are $S(-2) = \langle (1,-2,0), (0,0,1) \rangle$ and $S(3) = \langle (2,1,0) \rangle$.

(b) Q is indefinite, since there are eigenvalues of opposite sign. Restricting Q to $y = -2x$ the following two dimensional quadratic form obtains

$$\begin{pmatrix} x & -2x & z \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ -2x \\ z \end{pmatrix} = 2x^2 - (-2x)^2 - 2z^2 + 4(-2x^2) = -10x^2 - 2z^2,$$

which is clearly negative definite.

$\boxed{3}$

Consider the plane region

$$S = \{(x, y) \in \mathbb{R}^2 \,:\, y \geq x^2 - 3x + 2,\ y \leq 2\}.$$

(a) (10 points) Draw S and calculate its area as a double integral.

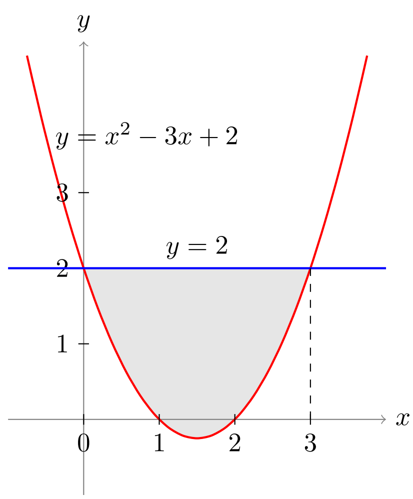
(b) (10 points) Claculate

$$\iint_S x \, dxdy,$$

where S is the region considered above.

Solution:

(a) The region S is represented below. Note that $x^2 - 3x + 2 \leq y \leq 2$ if and only if $0 \leq x \leq 3$.



The area of S is the value of the integral

$$\iint_S 1 \, dxdy = \int_0^3 \left(\int_{x^2-3x+2}^{2} 1 \, dy \right) dx = \int_0^3 (\cancel{2} - x^2 + 3x - \cancel{2}) \, dx = -\frac{x^3}{3} + 3\frac{x^2}{2} \Big|_{x=0}^{x=3} = -9 + \frac{27}{2} = \frac{9}{2}.$$

(b)

$$\iint_S x \, dxdy = \int_0^3 x \left(\int_{x^2-3x+2}^{2} dy \right) dx = \int_0^3 x(\cancel{2} - x^2 + 3x - \cancel{2}) \, dx = -\frac{x^4}{4} + \cancel{3}\frac{x^3}{\cancel{3}} \Big|_{x=0}^{x=3} = -\frac{81}{4} + 27 = \frac{27}{4}.$$

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(a) (10 points) Calculate the integral

$$\int_1^\infty \frac{1}{x^3+x}\,dx.$$

Hint: decompose $\frac{1}{x^3+x} = \frac{1}{x(x^2+1)}$ into simple fractions and then use $\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)$ and $a\ln b = \ln(b^a)$.

(b) (10 points) The continuous function $f : [0,\infty) \to (0,1]$ satisfies

$$\int_0^x \frac{t}{f(t)}\,dt = \int_0^{x^2} e^{t^2}\,dt$$

for all $x \geq 0$. Find $f(x)$. Hint: use the Fundamental Theorem of Calculus (or the generalization, the Leibniz Rule).

Solution:

(a) Decomposing the integrand into simple fractions, we obtain

$$\frac{1}{x^3+x} = \frac{1}{x(x^2+1)} = \frac{1}{x} - \frac{x}{x^2+1}.$$

Thus

$$\int \frac{1}{x^3+x}\,dx = \int \frac{1}{x} - \frac{x}{x^2+1}\,dx = \int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx.$$

Hence

$$\int \frac{1}{x^3+x}\,dx = \ln x - \frac{1}{2}\ln(x^2+1) = \ln\frac{x}{\sqrt{x^2+1}}.$$

Thus

$$\int_1^\infty \frac{1}{x^3+x}\,dx = \lim_{b\to\infty} \ln\frac{x}{\sqrt{x^2+1}}\Bigg|_{x=1}^{b} = \lim_{b\to\infty} \ln\frac{b}{\sqrt{b^2+1}} - \ln\frac{1}{\sqrt{1+1}} = \ln 1 - \ln\frac{1}{\sqrt{2}} = -\ln\frac{1}{\sqrt{2}} > 0.$$

(b) Deriving both sides of the equality

$$\int_0^x \frac{t}{f(t)}\,dt = \int_0^{x^2} e^{t^2}\,dt$$

we get

$$\frac{x}{f(x)} = 2xe^{x^4}, \quad \text{thus} \quad f(x) = \frac{1}{2}e^{-x^4}.$$

$\boxed{5}$

(a) (10 points) Consider the sequence $\{x_n\}_{n=1}^{\infty}$, which satisfies $x_1 = -6$ and $x_{n+1} = \frac{2}{3}x_n + 3$, for all $n \geq 1$.

(i) Prove that the sequence is increasing.

(ii) Prove that the sequence is bounded between -6 and 12, that is, $-6 \leq x_n \leq 12$ for all n

(iii) Deduce that the sequence is convergent and find the limit.

(b) (10 points) Prove that the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}.$$

is convergent. If we add the first four terms of the series, what is the maximum error if we take the sum of the infinite series as the sum of the first four terms? Is this error by excess or by default?

Solution:

(a) The sequence converges to 9.

Note that $x_2 = 1 > -6 = x_1$. If we suppose that $x_{n-1} < x_n$, then

$$x_n = \frac{2}{3}x_{n-1} + 3 < \frac{2}{3}x_n + 3 = x_{n+1},$$

and by the Induction Principle, the sequence is increasing.

Let us see that $-6 \leq x_n \leq 12$ for all n. It holds for $n = 1$; suppose that it is true for n and let us see that it is true for $n + 1$.

$$x_{n+1} = \frac{2}{3}x_n + 3 \geq \frac{2}{3}(-6) + 3 = -1 > -6$$

and

$$x_{n+1} = \frac{2}{3}x_n + 3 \leq \frac{2}{3}(12) + 3 = 11 < 12,$$

hence, by the Induction Principle, the sequence is bounded between -6 and 12.

Since that the sequence is increasing and bounded, it is convergent to a number L. L solves the equation

$$L = \frac{2}{3}L + 3 \ \Rightarrow \ L = 9.$$

(b) It is an alternate series that converges by the Theorem of Leibniz (monotone converging to 0)

$S_4 = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16}$ approximates the value of the infinite sum by default since the series is truncated in an even number of terms. The maximum error is $x_5 = \frac{1}{25} = 0,04$.