UC3M
Mathematics for Economics II (final exam)
June 1st, 2022 **SOLVED**

1

(a) (10 points) It is known that

$$\begin{vmatrix} 1 & 0 & 1 & a \\ 2 & -1 & 1 & b \\ 3 & 4 & 2 & c \\ 1 & -1 & 1 & d \end{vmatrix} = 0.$$

Solve for x in the equation

$$\begin{vmatrix} 1 & 0 & 1 & a \\ 2 & -1 & 1 & b \\ 3 & 4 & 2 & c \\ 1 & -1 & 1 & x+d \end{vmatrix} = -10.$$

(b) (10 points) Discuss and solve the linear system

$$\begin{cases} y+z &= 1 \\ 2x-y-z &= -1 \\ -4x+5y+5z &= 5 \end{cases}$$

Solution:

(a) By the properties of the determinant

$$\begin{vmatrix} 1 & 0 & 1 & a \\ 2 & -1 & 1 & b \\ 3 & 4 & 2 & c \\ 1 & -1 & 1 & x+d \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 \\ 3 & 4 & 2 & 0 \\ 1 & -1 & 1 & x \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 & a \\ 2 & -1 & 1 & b \\ 3 & 4 & 2 & c \\ 1 & -1 & 1 & d \end{vmatrix}.$$

Hence, since the second determinant at the right of the above expression is zero by hypothesis, the equation becomes

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 \\ 3 & 4 & 2 & 0 \\ 1 & -1 & 1 & x \end{vmatrix} = -10.$$

Expanding the determinant by the fourth column it transforms into

$$x \begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 3 & 4 & 2 \end{vmatrix} = -10$$

which reduces to $5x = -10$, thus $x = -2$.

(b) It is easy to see that the determinant of the system is 0 and that both the matrix of the system and the augmented matrix have rank 2, hence the system admits many solutions, which depend on a single parameter.

In fact, we can proceed directly to find the solutions: from the first equation we get $y = 1 - z$ and then from the second equation we obtain $x = 0$. The third equation is trivially satisfied with $x = 0$ and $y = 1 - z$, thus if we choose parameter $z = t$, then the solutions are given by $x = 0$, $y = 1 - t$, $z = t$, with $t \in \mathbb{R}$.

2

Consider the following matrix with parameter $a \neq 0$.

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & a \\ 0 & \frac{3}{a} & 4 \end{pmatrix}.$$

(a) (10 points) For what values of the parameter a is the matrix A diagonalizable? Justify your answer.

(b) (10 points) For the values of the parameter a for which the matrix A is diagonalizable, find a matrix P and a diagonal matrix D associated to A. Justify your answer.

Solution:

(a) We calculate $p_A(\lambda) = |A - \lambda I|$.

$$\begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & 2-\lambda & a \\ 0 & \frac{3}{a} & 4-\lambda \end{vmatrix} = (5-\lambda)\left((2-\lambda)(4-\lambda)-3\right) = (5-\lambda)(\lambda^2 - 6\lambda + 5).$$

Hence, the eigenvalues are $\lambda = 5$ (double) and $\lambda = 1$ (simple).

Matrix A is diagonalizable iff $\text{rank}(A - 5I_3) = 1$. Clearly,

$$A - 5I_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & a \\ 0 & \frac{3}{a} & -1 \end{pmatrix},$$

is of rank 1, thus A is diagonalizable for all $a \neq 0$.

(b)

- The eigenspace associated to $\lambda = 5$ is given by all solutions of the homogeneous system $(A - 5I_3)\mathbf{x} = \mathbf{0}$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & a \\ 0 & \frac{3}{a} & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

  This system reduces to the single equation $-3y + az = 0$. Letting x, z as parameters, the solutions are given by

$$x(1,0,0) + z(0, \frac{a}{3}, 1).$$

- The eigenspace associated to $\lambda = 1$ is given by all solutions of the homogeneous system $(A - I_3)\mathbf{x} = \mathbf{0}$:

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & a \\ 0 & \frac{3}{a} & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

  This is a system with two independent equations, $4x = 0$ and $y + az = 0$. The solutions are described by

$$z(0, -a, 1), \qquad z \in \mathbb{R}.$$

The matrices requested are

$$D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{a}{3} & -a \\ 0 & 1 & 1 \end{pmatrix}.$$

3

(a) (10 points) Classify the quadratic form $Q(x, y, z) = x^2 + 2y^2 + az^2 + 2xy + 2xz + 4yz$, where $a \in \mathbb{R}$ is a parameter.

(b) (10 points) Find the value of the double integral

$$\iint_D xe^{xy}\, dxdy,$$

where $D = [0, 2] \times [0, 1]$.

Solution:

(a) The matrix associated to the quadratic form is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & a \end{pmatrix},$$

with principal minors $A_1 = 1 > 0$, $A_2 = 1 > 0$ and $A_3 = |A| = a - 2$, Thus, Q is indefinite if $a < 2$, positive semidefinite if $a = 2$ and positive definite if $a > 2$.

(b)

$$\iint_D xe^{xy}\, dxdy = \int_0^2 dx \int_0^1 xe^{xy}\, dy = \int_0^2 e^{xy}\Big|_{y=0}^{y=1} dx = \int_0^2 e^x - 1\, dx = e^x - x\Big|_{x=0}^{x=2} = e^2 - 3.$$

4

(a) (10 points) Study the convergence of the improper integral

$$\int_2^\infty \frac{1}{x^2} \sin\left(\frac{1}{x}\pi\right) dx$$

and find its value if it results to be convergent.

(b) (10 points) Let the function $F : [0,\infty) \longrightarrow \mathbb{R}$ defined by means of the integral

$$F(t) = \int_0^t x\,(t-x)\,dx, \quad \text{for } t \geq 0.$$

Find $F'(t)$.

Solution:

(a) We compute directly the integral. Let $b > 2$.

$$\begin{aligned}\int_2^b \frac{1}{x^2} \sin\left(\frac{1}{x}\pi\right) dx &= \frac{1}{\pi}\cos\left(\frac{1}{x}\pi\right)\Big|_2^b \\ &= \frac{1}{\pi}\left(\cos\left(\frac{1}{b}\pi\right) - \cos\left(\frac{1}{2}\pi\right)\right) \\ &= \frac{1}{\pi}\cos\left(\frac{1}{b}\pi\right),\end{aligned}$$

since $\cos\left(\frac{1}{2}\pi\right) = 0$. Thus

$$\begin{aligned}\int_2^\infty \frac{1}{x^2} \sin\left(\frac{1}{x}\pi\right) dx &= \lim_{b\to\infty} \int_2^b \frac{1}{x^2} \sin\left(\frac{1}{x}\pi\right) dx \\ &= \lim_{b\to\infty} \frac{1}{\pi}\cos\left(\frac{1}{b}\pi\right) \\ &= \frac{1}{\pi}\cos 0 = \frac{1}{\pi},\end{aligned}$$

since $\cos 0 = 1$.

(b) We can compute directly $F(t)$ by integration and then derive to find $F'(t)$.

$$F(t) = \int_0^t tx\,dx - \int_0^t x^2\,dx = \frac{1}{2}tx^2\Big|_{x=0}^{x=t} - \frac{1}{3}x^3\Big|_{x=0}^{x=t} = \frac{1}{2}t^3 - \frac{1}{3}t^3 = \frac{1}{6}t^3.$$

Thus, $F'(t) = \frac{1}{2}t^2$.

Other possibility is to apply Leibniz's rule directly:

$$F'(t) = \cancel{t(t-t)} + \int_0^t \frac{\partial}{\partial t}\left(x(t-x)\right)\,dx = \int_0^t x\,dx = \frac{1}{2}x^2\Big|_0^t = \frac{1}{2}t^2.$$

5

(a) (10 points) Consider the sequence $\{x_n\}_{n=1}^{\infty}$, which satisfies $x_1 = 10$ and $x_{n+1} = \frac{1}{2}x_n + \frac{1}{5}$, for $n \geq 1$. Prove that the sequence is convergent and find its limit. Hint: prove that the sequence is monotone decreasing and bounded.

(b) (10 points) Study the character of the series

$$\sum_{n=1}^{\infty} \frac{\left(\frac{n}{10}\right)^n}{n!}.$$

Solution:

(a) Note that $x_1 = 10 > 5 + \frac{1}{5} = x_2$. Assuming that $x_{n-1} > x_n$ we have

$$x_n = \frac{1}{2}x_{n-1} + \frac{1}{5} > \frac{1}{2}x_n + \frac{1}{5} = x_{n+1},$$

thus by the induction principle, the sequence is monotone decreasing.

On the other hand, $0 \leq x_1 \leq 10$. Assuming that $0 \leq x_n \leq 10$, we have

$$x_{n+1} = \frac{1}{2}x_n + \frac{1}{5} \geq \frac{1}{5} \geq 0 \quad \text{and} \quad x_{n+1} = \frac{1}{2}x_n + \frac{1}{5} \leq 5 + \frac{1}{5} \leq 10,$$

thus by the induction principle, the sequence is bounded between 0 and 10.

Since that the sequence is both monotone and bounded, it converges, to a limit L. To find L we have to solve the equation

$$L = \frac{1}{2}L + \frac{1}{5} \Rightarrow L = \frac{2}{5}.$$

(b) It is a series of positive terms, $a_n = \frac{\left(\frac{n}{10}\right)^n}{n!}$. Using the ratio test, we get, after some rearranging and simplification

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}10^n n!}{n^n 10^{n+1}(n+1)!} = \left(\frac{n+1}{n}\right)^n \frac{1}{10} \to \frac{e}{10} < 1 \quad \text{as } n \to \infty.$$

Thus, the series converges.