UC3M
Mathematics for Economics II (final exam)
June 29, 2022 **Solutions**

1

(a) (10 points) It is known that

$$\begin{vmatrix} 1 & 0 & 1 & a \\ 2 & -1 & 1 & b \\ 3 & 4 & 0 & c \\ 1 & -1 & 0 & d \end{vmatrix} = 10.$$

Find the value of the determinant

$$\begin{vmatrix} 1 & 0 & 2a+6 & 1 \\ 2 & -1 & 2b & 1 \\ 3 & 4 & 2c & 0 \\ 1 & -1 & 2d & 0 \end{vmatrix}.$$

(b) (10 points) Find an echelon matrix equivalent to the 4×5 matrix A below.

$$A = \begin{pmatrix} 1 & -1 & 2 & 2 & 1 \\ 3 & 3 & 0 & 2 & -1 \\ 2 & -4 & 6 & 1 & -1 \\ -1 & -3 & 2 & -3 & -2 \end{pmatrix}$$

Use the echelon matrix to calculate the rank of A.

Solution:

(a) By the properties of the determinant

$$\begin{vmatrix} 1 & 0 & 2a+6 & 1 \\ 2 & -1 & 2b & 1 \\ 3 & 4 & 2c & 0 \\ 1 & -1 & 2d & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 1 & 2a+6 \\ 2 & -1 & 1 & 2b \\ 3 & 4 & 0 & 2c \\ 1 & -1 & 0 & 2d \end{vmatrix} = - \left(\begin{vmatrix} 1 & 0 & 1 & 2a \\ 2 & -1 & 1 & 2b \\ 3 & 4 & 0 & 2c \\ 1 & -1 & 0 & 2d \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 & 6 \\ 2 & -1 & 1 & 0 \\ 3 & 4 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{vmatrix} \right)$$

$$= -2 \begin{vmatrix} 1 & 0 & 1 & a \\ 2 & -1 & 1 & b \\ 3 & 4 & 0 & c \\ 1 & -1 & 0 & d \end{vmatrix} - (-6) \begin{vmatrix} 2 & -1 & 1 \\ 3 & 4 & 0 \\ 1 & -1 & 0 \end{vmatrix} = -20 - (-6) \times (-7) = -62.$$

(b) We use Gaussian elimination to find an echelon form equivalent to the given matrix.

- row 2−3×(row 1), row 3−2×row 1), and row 4+row 1.

$$\begin{pmatrix} 1 & -1 & 2 & 2 & 1 \\ 0 & 6 & -6 & -4 & -4 \\ 0 & -2 & 2 & -3 & -3 \\ 0 & -4 & 4 & -1 & -1 \end{pmatrix}$$

Exchange rows.

$$\begin{pmatrix} 1 & -1 & 2 & 2 & 1 \\ 0 & -2 & 2 & -3 & -3 \\ 0 & -4 & 4 & -1 & -1 \\ 0 & 6 & -6 & -4 & -4 \end{pmatrix}$$

- Subtract row 2 multiplied by 2 to row 3, and add row 2 multiplied by 3 to row 4.

$$\begin{pmatrix} 1 & -1 & 2 & 2 & 1 \\ 0 & -2 & 2 & -3 & -3 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & -13 & -13 \end{pmatrix}$$

- We arrive to the echelon form where only the bottom row is null, thus the rank of the matrix is 3.

$$\begin{pmatrix} 1 & -1 & 2 & 2 & 1 \\ 0 & -2 & 2 & -3 & -3 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

2

Consider the following matrix.

$$A = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}.$$

(a) (10 points) Calculate de eigenvalues and eigenvectors of A and prove that A is diagonalizable. Find a diagonal form D of A and an associated matrix P. Justify your answers.

(b) (10 points) Using the results obtained in part (a) above, calculate the nth–power of A, that is, calculate

$$\begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}^n,$$

where n is an arbitrary positive integer. Justify your answer.

Solution:

(a) $p_A(\lambda) = |A - \lambda I| = (2/3 - \lambda)^2 - 1/9$. The roots (eigenvalues) are $\lambda = 1$ and $\lambda = 1/3$, thus A is diagonalizable since all the eigenvalues are real and distinct.

The eigenspace $S(1)$ is obtained from solving $\begin{cases} -(1/3)x - (1/3)y & = 0 \\ -(1/3)x - (1/3)y & = 0 \end{cases}$, that is, $S(1) = \{(x, -x) : x \in \mathbb{R}\}$, thus $(1, -1)$ is an eigenvalue associated to $\lambda = 1$.

The eigenspace $S(1/3)$ is obtained from solving $\begin{cases} (1/3)x - (1/3)y & = 0 \\ -(1/3)x + (1/3)y & = 0 \end{cases}$, that is, $S(1/3) = \{(x, x) : x \in \mathbb{R}\}$, thus $(1, 1)$ is an eigenvalue associated to $\lambda = 1/3$.

Hence

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

(b) From $D = P^{-1}AP$ we obtain $A = PDP^{-1}$ and then $A^n = PD^nP^{-1}$ as it is shown in the class notes.

The inverse of P is $P^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$.

Thus

$$\begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}^n = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix}^n \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix},$$

obtaining

$$\begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}^n = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/3^n \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

Finally

$$\begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}^n = \begin{pmatrix} 1/2 + (1/2)(1/3^n) & -1/2 + (1/2)(1/3^n) \\ -1/2 + (1/2)(1/3^n) & 1/2 + (1/2)(1/3^n) \end{pmatrix}$$

3

(a) (10 points) Classify the quadratic form $Q(x,y,z) = \frac{a}{3}x^2 + ay^2 + \frac{27}{a}z^2 + axy + 3yz$, where $a \neq 0$ is a parameter.

(b) (10 points) Draw the plane set $D = \{(x,y) : 0 \le x \le 2,\ y \le x,\ y \le 2-x\}$ and find the value of the double integral

$$\iint_D \sqrt{x+y}\, dx\, dy.$$

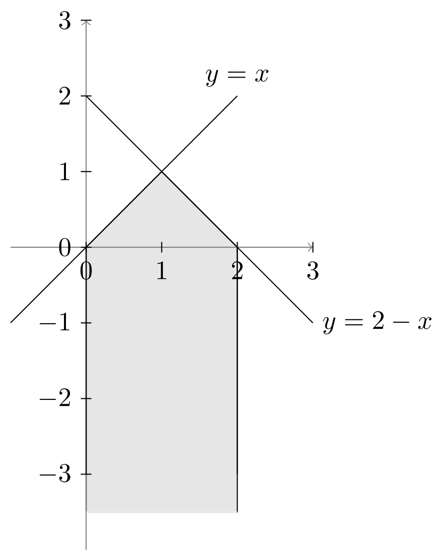
Solution:

(a) The matrix associated to the quadratic form is

$$A = \begin{pmatrix} \frac{a}{3} & \frac{a}{2} & 0 \\ \frac{a}{2} & a & \frac{3}{2} \\ 0 & \frac{3}{2} & \frac{27}{a} \end{pmatrix},$$

with principal minors $A_1 = \frac{a}{3}$, $A_2 = \frac{a^2}{12}$ and $A_3 = |A| = \frac{3}{2}a$. Thus, Q is definite positive if $a > 0$ and definite negative if $a < 0$.

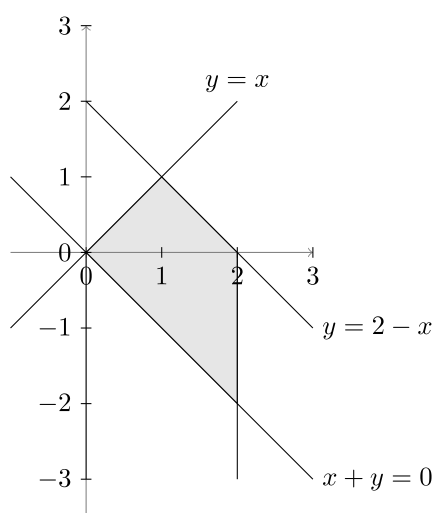
(b) The set D is the infinite region represented in the figure below



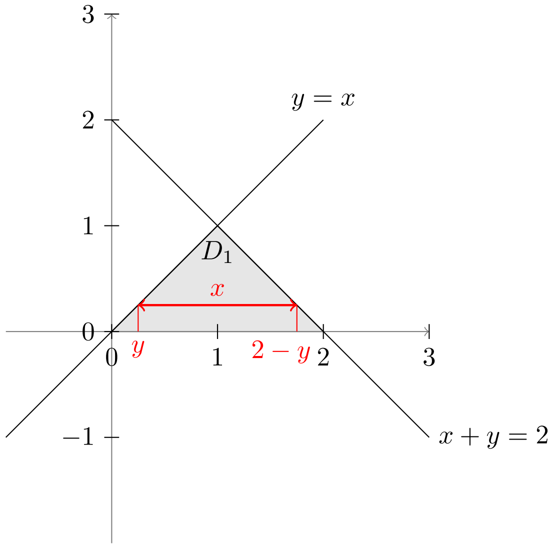
To calculate the integral, note that integrand function $\sqrt{x+y}$ is defined only when $x+y \ge 0$, thus the effective region of integration is the subset of D given by

$$D' = \{(x,y) : 0 \le x \le 2,\ y \le x,\ y \le 2-x, x+y \ge 0\}.$$

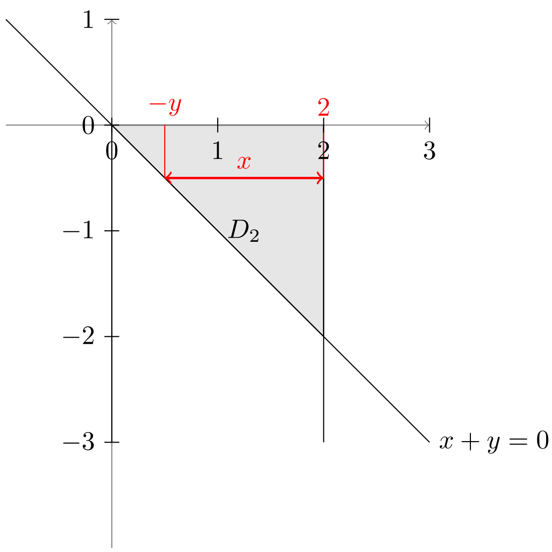
The subset D' is represented here



To integrate, we write $D' = D_1 \cup D_2$, where D_1 is the triangle of vertexes $(0,0)$, $(1,1)$ and $(2,0)$ and D_2 is the triangle of vertexes $(0,0)$, $(2,-2)$ and $(2,0)$.



$$\begin{aligned}
\iint_{D_1} \sqrt{x+y}\,dx\,dy &= \int_0^1 dy \int_y^{2-y} \sqrt{x+y}\,dx \\
&= \int_0^1 \frac{2}{3}(x+y)^{3/2}\Big|_{x=y}^{x=2-y} dy \\
&= \frac{2}{3}\int_0^1 \left(2^{3/2} - (2y)^{3/2}\right) dy \\
&= \frac{2}{3}2^{3/2}\int_0^1 \left(1 - y^{3/2}\right) dy \\
&= \frac{2}{3}2^{3/2}\left(y\Big|_{y=0}^{y=1} - \frac{2}{5}y^{5/2}\Big|_{y=0}^{y=1}\right) \\
&= \frac{2}{3}2^{3/2}\left(1 - \frac{2}{5}\right) \\
&= \frac{2\sqrt{8}}{5}\left(= \frac{4\sqrt{2}}{5}\right).
\end{aligned}$$



$$\begin{aligned}
\iint_{D_2} \sqrt{x+y}\,dx\,dy &= \int_{-2}^0 dy \int_{-y}^{2} \sqrt{x+y}\,dx \\
&= \int_{-2}^0 \frac{2}{3}(x+y)^{3/2}\Big|_{x=-y}^{x=2} dy \\
&= \frac{2}{3}\int_{-2}^0 \left((2+y)^{3/2} - \cancel{(-y+y)^{3/2}}\right) dy \\
&= \frac{2}{3}\int_{-2}^0 (2+y)^{3/2}\,dy \\
&= \frac{2}{3}\frac{2}{5}(2+y)^{5/2}\Big|_{y=-2}^{y=0} \\
&= \frac{4}{15}\left(2^{5/2} - \cancel{(2-2)^{5/2}}\right) \\
&= \frac{16}{15}\sqrt{2}.
\end{aligned}$$

Hence

$$\iint_D \sqrt{x+y}\,dx\,dy = \iint_{D_1} \sqrt{x+y}\,dx\,dy + \iint_{D_2} \sqrt{x+y}\,dx\,dy = \frac{4}{5}\sqrt{2} + \frac{16}{15}\sqrt{2} = \frac{28}{15}\sqrt{2}.$$

4

(a) (10 points) Study the convergence of the improper integral

$$\int_3^4 \ln(x-3)\,dx$$

and find its value if it results to be convergent.
Hint: Calculate $\lim_{x\to 3^+}(x-3)\ln(x-3)$ using L'Hopital's Rule.

(b) (10 points) Study the convergence of the improper integral

$$\int_5^\infty \frac{1}{25+x^2}\,dx$$

and find its value if it results to be convergent.

Solution:

(a) We first find a primitive of the integrand. Let $u = \ln(x-3)$ and $dv = dx$, so $du = \frac{1}{x-3}\,dx$ and $v = x$. By parts,

$$\int \ln(x-3)\,dx = x\ln(x-3) - \int \frac{x}{x-3}\,dx.$$

Now, $\frac{x}{x-3} = 1 + \frac{3}{x-3}$, hence

$$\int \frac{x}{x-3}\,dx = \int \left(1+\frac{3}{x-3}\right)dx = x + 3\ln(x-3) \text{ (plus an arbitrary constant that we do not consider).}$$

Hence

$$\int \ln(x-3)\,dx = x\ln(x-3) - x - 3\ln(x-3) = (x-3)\ln(x-3) - x.$$

Thus

$$\begin{aligned}
\int_3^4 \ln(x-3)\,dx &= \lim_{a\to 3^+}\int_a^4 \ln(x-3)\,dx \\
&= \lim_{a\to 3^+}\Big((x-3)\ln(x-3) - x\Big)\Big|_{x=a}^{4} \\
&= \lim_{a\to 3^+}\Big((4-3)\ln(4-3) - 4 - (a-3)\ln(a-3) - a\Big) \\
&= (4-3)\ln(4-3) - 4 - \lim_{a\to 3^+}\Big((a-3)\ln(a-3) - a\Big) \\
&= 1\times 0 - 4 + 3 = -1,
\end{aligned}$$

once we have calculated that $\lim_{x\to 3^+}(x-3)\ln(x-3) = 0$ using the hint:

$$\lim_{x\to 3^+}(x-3)\ln(x-3) = \lim_{x\to 3^+}\frac{\ln(x-3)}{\frac{1}{x-3}} = \lim_{x\to 3^+}\frac{\frac{1}{x-3}}{-\frac{1}{(x-3)^2}} = \lim_{x\to 3^+} -\frac{(x-3)^{\not 2}}{\cancel{(x-3)}} = 0.$$

(b) Since $25 + x^2 \geq x^2$, we have $\frac{1}{25+x^2} \leq \frac{1}{x^2}$. Since $\int_5^\infty \frac{1}{x^2}\,dx$ is convergent, we have, by the comparison principle, that the requested integral converges. Let us find its value.

$$\begin{aligned}
\int_5^\infty \frac{1}{25+x^2}\,dx &= \frac{1}{25}\int_5^\infty \frac{1}{1+(x/5)^2}\,dx \\
&= \frac{1}{25}\int_1^\infty \frac{5}{1+t^2}\,dt \quad (t = \tfrac{x}{5}, dt = \tfrac{dx}{5}) \\
&= \frac{5}{25}\int_1^\infty \frac{1}{1+t^2}\,dt \\
&= \frac{1}{5}\arctan t\Big|_{t=0}^{t=1} = \frac{1}{5}\Big(\arctan 1 - \arctan 0\Big) = \frac{1}{5}\frac{\pi}{4} = \frac{\pi}{20}.
\end{aligned}$$

5

(a) (10 points) Calculate the following limit

$$\lim_{n\to\infty} \frac{\sqrt{n^2+7}-n}{\sqrt{n^2+13}-n}.$$

Justify your answer.

(b) (10 points) State the Theorem of Leibniz about alternating series.
Check that the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt[3]{n}}.$$

fulfils all the hypotheses of the Theorem of Leibniz. If S is the sum of the above series, and S_{26} denotes the partial sum of the first 26 terms of the series,

$$\frac{1}{\sqrt[3]{1}} - \frac{1}{\sqrt[3]{2}} + \cdots + \frac{1}{\sqrt[3]{25}} - \frac{1}{\sqrt[3]{26}},$$

which is the bound to $|S - S_{26}|$ provided by the Theorem of Leibniz?

Solution:

(a)

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sqrt{n^2+7}-n}{\sqrt{n^2+13}-n} &= \lim_{n\to\infty} \frac{(\sqrt{n^2+7}-n)(\sqrt{n^2+7}+n)(\sqrt{n^2+13}+n)}{(\sqrt{n^2+13}-n)(\sqrt{n^2+13}+n)(\sqrt{n^2+7}+n)} \\
&= \lim_{n\to\infty} \frac{7(\sqrt{n^2+13}+n)}{13(\sqrt{n^2+7}+n)} \\
&= \frac{7}{13}\lim_{n\to\infty} \frac{\sqrt{1+\frac{13}{n^2}}+1}{\sqrt{1+\frac{7}{n^2}}+1} \\
&= \frac{7}{13}.
\end{aligned}$$

(b) For the statement of the Theorem of Leibniz for alternating series, see the class notes.
Let us denote $a_n = \frac{1}{\sqrt[3]{n}}$. Then $|S - S_{26}| < a_{27} = \frac{1}{\sqrt[3]{27}} = \frac{1}{3}$.