UC3M
Mathematics for Economics II (final exam)
May 25th 2021 **SOLVED**

$\boxed{1}$

Consider the following system of linear equations with parameter $a \in \mathbb{R}$.

$$\left.\begin{array}{rcl} x+z+t & = & 5 \\ x+y+2z+2t & = & 6 \\ x+z+(a+2)t & = & 8 \end{array}\right\}$$

(a) (10 points) Discuss the type of system according to the values of parameter a.

(b) (10 points) Solve the system when $a = 2$. In this case find all solutions that satisfy that $x = 1$.

Solution:

(a) We consider the augmented matrix

$$A^* = \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 5 \\ 1 & 1 & 2 & 2 & 6 \\ 1 & 0 & 1 & a+2 & 8 \end{array}\right)$$

by Gauss operations $row2 - row1$ and $row3 - row1$ we get

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 5 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & a+1 & 3 \end{array}\right)$$

If $a \neq -1$ then $rank(A) = rank(A^*) = 3 <$ number of unknows and from the Rouché-Fröbenius theorem the system is consistent and underdetermined. If $a = -1$ then $rank(A) = 2 \neq rank(A^*) = 3$ and from the Rouché-Fröbenius theorem the system is inconsistent.

(b) When $a = 2$ we are under the first case, consistent and underdetermined, since there are 4 unknows and the rank is 3, one of the them becomes a free variable. The augmented matrix becomes

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 5 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 & 3 \end{array}\right)$$

and the system is

$$\left.\begin{array}{rcl} x+z+t & = & 5 \\ y+z+t & = & 1 \\ 3t & = & 3 \end{array}\right\}$$

Noticing that $t = 1$, then the choices for the free variable are x, y and z. Choosing, for instance z, the system becomes

$$\left.\begin{array}{rcl} x+t & = & 5-z \\ y+t & = & 1-z \\ 3t & = & 3 \end{array}\right\}$$

so the solutions of the system are

$$\left.\begin{array}{rcl} x & = & 4-z \\ y & = & -z \\ t & = & 1 \end{array}\right\} \text{ or } (4-z, -z, z, 1)$$

Finally, if $x = 1$ then from the first equation we have that $z = 3$ and from the second that $y = -3$ so there is only one solution that satisfies the condition: $x = 1$, $y = -3$, $z = 3$ and $t = 1$ or $(1, -3, 3, 1)$.

$\boxed{2}$

Consider the following matrix with parameters α y β.

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \alpha & 1 & -\frac{1}{2} \\ \beta & -\frac{1}{2} & 1 \end{pmatrix}.$$

(a) (10 points) For what values of the parameters α and β is the matrix A diagonalizable? Justify your answer.

(b) (10 points) For the values of the parameters α and β for which the matrix A is diagonalizable, find the matrix P and the diagonal matrix D associated to A. Justify your answer.

Solution:

(a) Expanding the determinant $|A - \lambda I|$ by the first row, we obtain

$$\begin{vmatrix} \frac{1}{2} - \lambda & 0 & 0 \\ \alpha & 1-\lambda & -\frac{1}{2} \\ \beta & -\frac{1}{2} & 1-\lambda \end{vmatrix} = \left(\frac{1}{2} - \lambda\right)\left((1-\lambda)^2 - \frac{1}{4}\right).$$

Hence, the eigenvalues are $\lambda = \frac{1}{2}$ (double) and $\lambda = \frac{3}{2}$ (simple).
Matrix A is diagonalizable iff $\operatorname{rank}(A - \frac{1}{2}I) = 1$. Since

$$A - \frac{1}{2}I = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & \frac{1}{2} & -\frac{1}{2} \\ \beta & -\frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

we have $\operatorname{rank}(A - \frac{1}{2}I) = 1$ iff $\beta = -\alpha$. Thus, A is diagonalizable iff this condition holds.

(b) By the above item, we suppose that $\beta = -\alpha$, that is, the matrix A is

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \alpha & 1 & -\frac{1}{2} \\ -\alpha & -\frac{1}{2} & 1 \end{pmatrix}$$

The eigenvectors associated to $\lambda = \frac{1}{2}$ are found by solving $(A - \frac{1}{2}I)u = 0$, that is

$$\begin{pmatrix} 0 & 0 & 0 \\ \alpha & \frac{1}{2} & -\frac{1}{2} \\ -\alpha & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system reduces to the single equation $\alpha x + \frac{1}{2}(y - z) = 0$. Two independent solutions are $(0, 1, 1)$ and$(1, 0, 2\alpha)$.
The eigenvectors associated to $\lambda = \frac{3}{2}$ are found by solving $(A - \frac{3}{2}I)u = 0$, that is

$$\begin{pmatrix} -1 & 0 & 0 \\ \alpha & -\frac{1}{2} & -\frac{1}{2} \\ -\alpha & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The system admits $(0, 1, -1)$ as a non-trivial solution.
Hence, assuming $\alpha + \beta = 0$, we have found

$$D = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 2\alpha & -1 \end{pmatrix}.$$

3

(a) (10 points) Classify the quadratic form $Q(x,y,z) = -2x^2 - y^2 - 8z^2 + 2xy - 4yz$.

(b) (10 points) Represent the plane set $D = \{(x,y) \in \mathbb{R}^2 : -x^2 \leq y \leq x^2, -1 \leq x \leq 1\}$ and calculate the double integral

$$\iint_D (x^2 - y) dx dy.$$

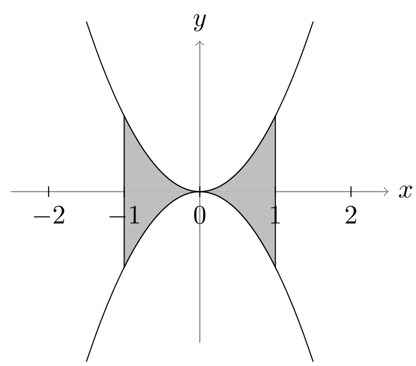
Solution:

(a) The matrix associated to the quadratic form is

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -1 & -2 \\ 0 & -2 & -8 \end{pmatrix}.$$

The lead minors are -2, 1 and 0, hence the quadratic form is negative semidefinite.

(b) The set D is the shadow region below.



$$\iint_D (x^2 - y) dx dy = \int_{-1}^{1} dx \int_{-x^2}^{x^2} (x^2 - y) dy = \int_{-1}^{1} \left[x^2 y - \frac{y^2}{2} \right]_{-x^2}^{x^2} dx = 2\int_{-1}^{1} x^4 dx = \frac{4}{5}.$$

$\boxed{4}$

(a) (10 points) Calculate the value of the integral

$$\int_0^6 \frac{2x}{(x^2-4)^{\frac{2}{3}}}\,dx.$$

(b) (10 points) Say whether the following improper integral is convergent or divergent depending on the values of the parameter $\gamma \geq 0$. In the cases where the integral is convergent, calculate its value.

$$\int_0^\infty xe^{-\gamma x}\,dx.$$

Solution:

(a) This is an improper integral of the second kind, since the integrand is not bounded at $x = 2$. Note that an antiderivative of the integrand in any interval avoiding the point 2 is immediate to obtain: $3\sqrt[3]{x^2-4}$. Splitting the interval of integration $[0,6]$ into $[0,2]$ and $[2,6]$, we find

$$\begin{aligned}\int_0^6 \frac{2x}{(x^2-4)^{\frac{2}{3}}}\,dx &= \int_0^{2^-} \frac{2x}{(x^2-4)^{\frac{2}{3}}}\,dx + \int_{2^+}^6 \frac{2x}{(x^2-4)^{\frac{2}{3}}}\,dx = \lim_{b\to 2^-}\left[3\sqrt[3]{x^2-4}\right]_0^b + \lim_{a\to 2^+}\left[3\sqrt[3]{x^2-4}\right]_a^6 \\ &= 3\lim_{b\to 2^-}(\sqrt[3]{b^2-4} - \sqrt[3]{-4}) + 3\lim_{a\to 2^+}(\sqrt[3]{6^2-4} - \sqrt[3]{a^2-4}) = 3\sqrt[3]{4} + 3\sqrt[3]{32} = 9\sqrt[3]{4}.\end{aligned}$$

(b) It is an improper integral of the second type, since the interval of integration is unbounded (and the integrand is bounded on bounded intervals). If $\gamma = 0$, the integral is $\int_1^\infty x\,dx = \infty$ If $\gamma > 0$, then integrate by parts $u = x$, $dv = e^{-\gamma x}$, $du = dx$, $v = -\frac{1}{\gamma}e^{-\gamma x}$ to obtain

$$\begin{aligned}\int_0^b \frac{x}{e^{\gamma x}}\,dx &= -\left[\frac{x}{\gamma}e^{-\gamma x}\right]_0^b + \frac{1}{\gamma}\int_0^b e^{-\gamma x}\,dx \\ &= -\frac{b}{\gamma}e^{-\gamma b} - \frac{1}{\gamma^2}\left[e^{-\gamma x}\right]_0^b \\ &= -\frac{b}{\gamma}e^{-\gamma b} - \frac{1}{\gamma^2}(e^{-\gamma b} - 1).\end{aligned}$$

Letting $b \to \infty$ and given that $be^{-\gamma b} \to 0$, $e^{-\gamma b} \to 0$, the integral is convergent, and its value is $\frac{1}{\gamma^2}$.

$\boxed{5}$

(a) (10 points) Calculate

$$\lim_{n\to\infty} e^{(\sqrt{n^2+\pi n+2}-n)}.$$

(b) (10 points) Study the character of the series

$$\sum_{n=1}^{\infty} n^{-p}p^n, \quad \text{where } p > 0.$$

Solution:

(a) We can calculate

$$\lim_{n\to\infty}(\sqrt{n^2+\pi n+2}-n) = \lim_{n\to\infty}\frac{\cancel{n^2}+\pi n+2-\cancel{n^2}}{\sqrt{n^2+\pi n+2}+n} = \lim_{n\to\infty}\frac{\cancel{n}\left(\pi+\frac{2}{n}\right)}{\cancel{n}\left(\sqrt{1+\frac{\pi}{n}+\frac{2}{n^2}}+1\right)} = \frac{\pi}{2},$$

where we have multiplied numerator and denominator by the conjugate and then used that $\frac{1}{n}$, $\frac{1}{n^2}$ go to 0 as $n \to \infty$. Hence, the limit is asked for is $e^{\frac{\pi}{2}}$.

(b) It is a series of positive terms. Let us call $a_n = n^{-p}p^n$. Using the ratio test, we get

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{-p}p^{n+1}}{n^{-p}p^n} = \left(\frac{n}{n+1}\right)^p p \to p$$

as $n \to \infty$. Thus, the series converges if $0 < p < 1$ and diverges if $p > 1$. When $p = 1$, it is the harmonic series, divergent.