UC3M
Mathematics for Economics II (retake)
June 18th 2021 **SOLVED**

$\boxed{1}$

Consider the following system of linear equations with parameters $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$$\begin{cases} ax + ay + z &= 1 \\ x + y + az &= b \\ x + ay + z &= a \end{cases}$$

(a) (15 points) Discuss the type of system according to the values of parameters a and b.

(b) (5 points) Solve the system when $a = -1$.

Solution:

(a) We consider the augmented matrix after exchanging the top and the bottom lines.

$$A^* = \left(\begin{array}{ccc|c} 1 & 1 & a & b \\ 1 & a & 1 & a \\ a & a & 1 & 1 \end{array}\right)$$

By Gauss operations $row2 - row1$ and $row3 - a \times row1$ we get

$$\left(\begin{array}{ccc|c} 1 & 1 & a & b \\ 0 & a-1 & 1-a & a-b \\ 0 & 0 & 1-a^2 & 1-ab \end{array}\right)$$

If $a \neq -1$ and $a \neq 1$, then $\text{rank}(A) = \text{rank}(A^*) = 3 =$ number of unknows and from the Rouché-Fröbenius Theorem the system is consistent and determined.

If $a = -1$ and $b = -1$, then $\text{rank}(A) = 2 = \text{rank}(A^*) < 3$ and from the Rouché-Fröbenius Theorem the system is consistent and underdetermined.

If $a = -1$ and $b \neq -1$, then $\text{rank}(A) = 2 < 3 = \text{rank}(A^*)$ and from the Rouché-Fröbenius Theorem the system is inconsistent.

If $a = 1$ and $b = 1$, then $\text{rank}(A) = 1 = \text{rank}(A^*) < 3$ and from the Rouché-Fröbenius Theorem the system is consistent and underdetermined.

If $a = 1$ and $b \neq 1$, then $\text{rank}(A) = 1 < 2 = \text{rank}(A^*)$ and from the Rouché-Fröbenius Theorem the system is inconsistent.

(b) When $a = -1$, the system is consistent only if $b = -1$. In this case the system is underdetermined. The augmented matrix becomes

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and the system is ($z = t$ is taken as parameter)

$$\left.\begin{array}{rcl} x + y - t &=& -1 \\ -2y + 2t &=& 0 \end{array}\right\}$$

The solution are given by vectors $(-1, t, t)$, $t \in \mathbb{R}$.

$\boxed{2}$

Consider the following matrix with real parameters $\alpha \neq 0$ and β.

$$A = \begin{pmatrix} \beta & 0 & 0 \\ 1 & 0 & -\alpha \\ 0 & -\alpha & 0 \end{pmatrix}$$

(a) (10 points) For what values of the parameters $\alpha \neq 0$ and β is the matrix A diagonalizable? Justify your answer.

(b) (10 points) For the values of the parameters $\alpha \neq 0$ and β for which the matrix A is diagonalizable, find the matrix P and the diagonal matrix D associated to A. Justify your answer.

Hint: The condition $\alpha \neq 0$ is important for reducing the number of cases!

Solution:

(a) Expanding the determinant $|A - \lambda I|$ by the first row, we obtain

$$\begin{vmatrix} \beta - \lambda & 0 & 0 \\ 1 & -\lambda & -\alpha \\ 0 & -\alpha & -\lambda \end{vmatrix} = (\beta - \lambda)\left(\lambda^2 - \alpha^2\right).$$

Hence, the eigenvalues are $\lambda = \beta$, $\lambda = \alpha$ and $\lambda = -\alpha$. We consider three cases: (i) $\beta = \alpha$; (ii) $\beta = -\alpha$; (iii) $\beta \neq \alpha$ and $\beta \neq -\alpha$. In what follows, remember that $\alpha \neq 0$.

- In case (i), $\lambda = \alpha$ is a double eigenvalue, thus A is diagonalizable iff $\operatorname{rank}(A - \alpha I) = 1$. Since

$$A - \alpha I = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -\alpha & -\alpha \\ 0 & -\alpha & -\alpha \end{pmatrix},$$

  has rank 2 because $\alpha \neq 0$, A is not diagonalizable.

- In case (ii), $\lambda = -\alpha$ is a double eigenvalue, thus A is diagonalizable iff $\operatorname{rank}(A + \alpha I) = 1$. Since

$$A + \alpha I = \begin{pmatrix} 0 & 0 & 0 \\ 1 & \alpha & -\alpha \\ 0 & -\alpha & \alpha \end{pmatrix},$$

  has rank 2 because $\alpha \neq 0$, A is not diagonalizable.

- In case (iii) A's eigenvalues are distinct. Thus A is diagonalizable.

(b) By the above item, we suppose that $\alpha \neq \beta$ and $\alpha \neq -\beta$. The eigenvalues of A are α, $-\alpha$ and β.

- $S(\alpha)$. Generated by $(0, -1, 1)$.
- $S(-\alpha)$. Generated by $(0, 1, 1)$.
- $S(\beta)$. Generated by $(\alpha^2 - \beta^2, -\beta, \alpha)$.

3

(a) (10 points) Classify the quadratic form $Q(x,y,z) = 4x^2 + y^2 + z^2 - 2cxy - 4xz$, where c is a real parameter.

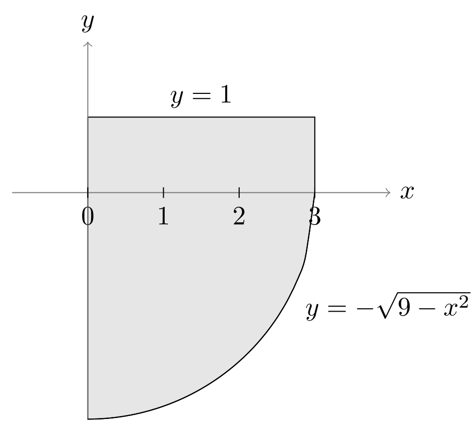
(b) (10 points) Draw the planar set $D = \{(x,y) \in \mathbb{R}^2 \, : \, -\sqrt{9-x^2} \le y \le 1, 0 \le x \le 3\}$ and calculate the double integral

$$\iint_D x dx\, dy.$$

Solution:

(a) The matrix associated to Q has leading minors $D_1 = 4$, $D_2 = 4 - c^2$ and $D_3 = -c^2$. Thus, Q is indefinite for all $c \neq 0$ and it is positive semidefinite if $c = 0$.

(b) The set D is the shadow region below.



$$\iint_D x\, dxdy = \int_0^3 x\, dx \int_{-\sqrt{9-x^2}}^1 dy = \int_0^3 \left[x + x\sqrt{9-x^2}\right] dx = \left[\frac{x^2}{2}\right]_0^3 - \left[\frac{1}{3}(9-x^2)^{\frac{3}{2}}\right]_0^3 = \frac{9}{2} + \frac{1}{3}(\sqrt{9})^3 = \frac{27}{2}.$$

$\boxed{4}$

(a) (10 points) Discuss the character of the improper integral

$$\int_1^e \frac{1}{x\ln x}\,dx.$$

In case that the integral is convergent, find its value.

(b) (10 points) Knowing that $\lim_{x\to\infty}\frac{x}{e^x}=0$, compute the value of the integral

$$\int_2^\infty (x-1)e^{-x}\,dx.$$

Solution:

(a) This is an improper integral of the second kind, since the integrand is not bounded at $x=1$ ($1/\ln 1 = 1/0 = \infty$.) Note that an antiderivative of the integrand in any interval $(1,b)$ is immediate to obtain: $\ln(\ln x)$. Hence (shortening the notation)

$$\int_1^e \frac{1}{x\ln x}\,dx = [\ln(\ln x)]_1^e = \ln 1 - \ln 0 = 0-(-\infty) = \infty.$$

The integral does not converge.

(b) It is an improper integral of the second type, since the interval of integration is unbounded (and the integrand is bounded on bounded intervals). Shortening the notation, we have

$$\int_2^\infty (x-1)e^{-x}\,dx = \left[(1-x)e^{-x}\right]_2^\infty + \int_2^\infty e^{-x}\,dx = \frac{1}{e^2} - \left[e^{-x}\right]_2^\infty = \frac{1}{e^2}+\frac{1}{e^2} = \frac{2}{e^2},$$

where we have taken parts $u=x-1$, $dv=e^{-x}dx$, $du=dx$, $v=-e^{-x}$ and used the hint provided in the question, that is, $xe^{-x}\to 0$ as $x\to\infty$.

$\boxed{5}$

(a) (10 points) It is known that the sequence of real numbers (x_n) satisfies the following properties:

- $x_1 = \frac{1}{2}$;
- $x_{n+1} = \sqrt{2x_n + 3}$;
- It is convergent to the real number L, that is, $\lim_{n\to\infty} x_n = L$.

Find the value of L.

(b) (10 points) Study the character of the series

$$\sum_{n=2}^{\infty}(a^{\frac{1}{2(n-1)}} - a^{\frac{1}{2n}}), \quad \text{where } a > 0.$$

When the series converges, find its value.
Hint: Note that it is a telescoping series.

Solution:

(a) Since the sequence (x_n) is convergent, the sequence $\sqrt{2x_n + 3}$ is also convergent. Moreover

$$\lim_{n\to\infty} \sqrt{2x_n + 3} = \sqrt{2 \lim_{n\to\infty} x_n + 3} = \sqrt{2L + 3},$$

thus

$$L = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \sqrt{2 \lim_{n\to\infty} x_n + 3} = \sqrt{2L + 3}.$$

Hence, L satisfies $L = \sqrt{2L+3}$, or $L^2 = 2L + 3$, $L^2 - 2L - 3 = 0$. Solving this quadratic equation we get two values, $L = -1$ and $L = 3$. But (x_n) is of positive terms, thus the limit cannot be -1. Thus the limit is $L = 3$.

(b) If $a = 1$, then the series is the null series. Suppose that $a > 0$ is not equal to 1. The series is telescoping, where consecutive terms from the second one cancel out. Hence, considering the partial sum of n terms we get

$$S_n = a^{\frac{1}{2}} \cancel{- a^{\frac{1}{4}} + a^{\frac{1}{4}}} \cancel{- a^{\frac{1}{6}} + a^{\frac{1}{6}}} - \cdots \cancel{- a^{\frac{1}{2(n-1)}} + a^{\frac{1}{2(n-1)}}} - a^{\frac{1}{2n}},$$

that is, $S_n = \sqrt{a} - a^{\frac{1}{2n}}$. Since $\lim_{n\to\infty} a^{\frac{1}{2n}} = a^{\frac{1}{\infty}} = a^0 = 1$, the sum of the series is $\sqrt{a} - 1$ for all $a > 0$.