UC3M
Mathematics for Economics II (mock final exam)

SOLVED

$\boxed{1}$

(a) (10 points) Consider the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -4 & -4 & 2 \\ 2 & 3 & 1 \end{pmatrix}.$$

Prove that A is diagonalizable, write its diagonal form and the matrix P.

(b) (10 points) Classify the quadratic form $Q(x,y,z) = 5x^2 + 2y^2 + 2z^2 - 4xy + 4xz + 8yz$ restricted to the subset $C = \{(x,y,z) \,:\, x+z=0\}$.

Solution:

(a) The characteristic polynomial of A is

$$p(\lambda) = (2-\lambda)(\lambda^2 + 3\lambda - 10).$$

The roots are 2, double and -5, simple.

The rank of $A - 2I$ is

$$\operatorname{rank}(A - 2I) = \operatorname{rank} \begin{pmatrix} 0 & 0 & 0 \\ -4 & -6 & 2 \\ 2 & 3 & -1 \end{pmatrix} = 1,$$

thus A is diagonalizable.

$S(2) = \langle (1,0,2), (0,1,3) \rangle$

$S(-5) = \langle (0,-2,1) \rangle$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 2 & 3 & 1 \end{pmatrix}.$$

(b) Easy.

$\boxed{2}$

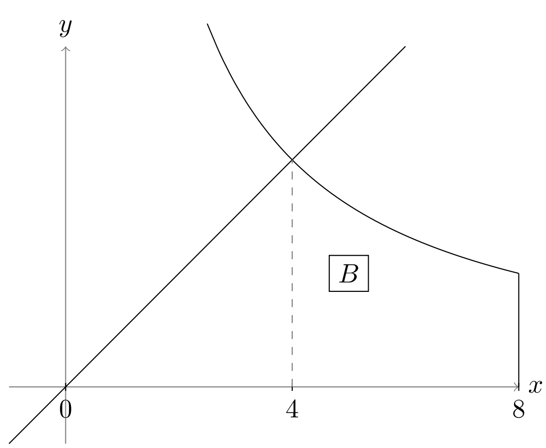
(a) (10 points) Let the set $B = \{(x,y) \,:\, 0 \le x \le 8,\ 0 \le y \le x,\ xy \le 16\}$. Make a drawing of this set.

(b) (10 points) Calculate the double integral

$$\iint_B x\,dx\,dy.$$

Solution:

(a)



(b)

$$\begin{aligned} \iint_B x\,dx\,dy &= \int_0^4 dx \int_0^x x\,dy + \int_4^8 dx \int_0^{\frac{16}{x}} x\,dy \\ &= \int_0^4 x^2\,dx + \int_4^8 16\,dx \\ &= \left.\frac{x^3}{3}\right|_0^4 + 16(8-4) = \frac{64}{3} + 64 = \frac{256}{3}. \end{aligned}$$

$\boxed{3}$

(a) (10 points) Calculate the area of the set A of $\mathbb{R}^2$ defined by $A = \{(x, y) : x \leq 0, 0 \leq y \leq \frac{1}{(4-x)^2}\}$.

(b) (10 points) Consider the improper integral

$$\int_0^4 \frac{xdx}{4 - x^2}.$$

Study whether it converges and, in this case, compute its value.

Solution:

(a) The area of A is the value of the improper integral

$$\int_{-\infty}^{0} \frac{dx}{(4-x)^2} = \lim_{a\to\infty} \int_{-a}^{0} \frac{dx}{(4-x)^2} = \lim_{a\to\infty} \left. \frac{1}{(4-x)} \right|_{-a}^{0} = \frac{1}{4}.$$

(b) $\frac{x}{4-x^2}$ is unbounded at $x = \pm 2$, thus the integral is improper. Choose to study $\int_0^2 \frac{xdx}{4-x^2}$. It is divergent.

$\boxed{4}$

(a) (10 points) The sequence $\{x_n\}_{n=1}^{\infty}$ satisfies $x_{n+1} = x_n^2 + 1$ for all $n = 1, 2, \ldots$ and $x_1 = \frac{1}{2}$. Justify why the sequence cannot be convergent.

(b) (10 points) Calculate $\lim_{n\to\infty} \left(\sqrt{n + 4n^2} - 2n\right)$.

Solution:

(a) By contradiction, if the sequence were convergent, call L to its limit. Then L satisfies $L = L^2 + 1$, a quadratic that has not real solutions. This is absurd, thus the sequence does not converge.

(b) The limit is $\frac{1}{\sqrt{4}+2} = \frac{1}{4}$.

$\boxed{5}$

(a) (10 points) Study the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{4^n n!}$.

(b) (10 points) Calculate the sum of the series $\sum_{n=1}^{\infty} \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right)$.

Solution:

(a) Let $a_n = \frac{n^n}{4^n n!}$. The ratio a_{n+1}/a_n reduces to

$$\frac{1}{4}\left(\frac{n+1}{n}\right)^n,$$

which converges to $e/4 < 1$ as $n \to \infty$, thus the series is convergent.

(b) This is a telescopic series $\sum_{n=1}^{\infty}(b_n - b_{n+1})$, with $b_n = \frac{1}{n!}$. Since $b_n \to 0$ as $n \to \infty$ and that $b_1 = \frac{1}{1!} = 1$, the sum is 1.