UC3M
Mathematics for Economics II (final exam)
May 27th 2019 **SOLVED**

$\boxed{1}$

Consider the linear system with parameters $\lambda \neq 0$ and $\mu \in \mathbb{R}$.

$$\begin{cases} x + \lambda y + \mu z &= 1 \\ x + \lambda\mu y + z &= \lambda \\ \mu x + \lambda y + z &= 1 \end{cases}$$

(a) (7 points) Discuss the system (recall that $\lambda \neq 0$).

(b) (3 points) Solve the system for the values $\lambda = \mu = -2$.

Solution:

(a) We will find an equivalent system in echelon form by applying the method of Gauss. We denote by A the matrix of the system and by A^* the augmented matrix. The system is equivalent to

$$\left(\begin{array}{ccc|c} 1 & \lambda & \mu & 1 \\ 0 & \lambda(\mu-1) & 1-\mu & \lambda-1 \\ 0 & -\lambda(\mu-1) & 1-\mu^2 & 1-\mu \end{array}\right),$$

after the operations $\text{row}_2 - \text{row}_1$, $\text{row}_3 - \mu\,\text{row}_1$. Now adding the second to the third row we obtain

$$\left(\begin{array}{ccc|c} 1 & \lambda & \mu & 1 \\ 0 & \lambda(\mu-1) & 1-\mu & \lambda-1 \\ 0 & 0 & 2-\mu-\mu^2 & \lambda-\mu \end{array}\right) \tag{1}$$

Note that $2-\mu-\mu^2 = (1-\mu)(2+\mu)$ and remember that we leave aside the case $\lambda = 0$.

- [Case 1] $\mu \neq 1$ and $\mu \neq -2$. Then $|A| \neq 0$, hence the system admits only one solution.
- [Case 2] $\mu = 1$. Then (1) becomes

$$\left(\begin{array}{ccc|c} 1 & \lambda & 1 & 1 \\ 0 & 0 & 0 & \lambda-1 \\ 0 & 0 & 0 & \lambda-1 \end{array}\right).$$

  Hence, A has rank 1 and A^* has rank 1 if $\lambda = 1$, but rank 2 if $\lambda \neq 1$. Thus, the system admits an infinite number of solutions if $\lambda = 1$, and no solution if $\lambda \neq 1$.
- [Case 3] $\mu = -2$. Then (1) becomes

$$\left(\begin{array}{ccc|c} 1 & \lambda & -2 & 1 \\ 0 & -3\lambda & 3 & \lambda-1 \\ 0 & 0 & 0 & \lambda+2 \end{array}\right). \tag{2}$$

  Hence, A has rank 2 and A^* has rank 2 if $\lambda = -2$, but rank 3 if $\lambda \neq -2$. Thus, the system admits an infinite number of solutions if $\lambda = -2$, and no solution if $\lambda \neq -2$.

(b) $\mu = -2$ corresponds to [Case 3] above. Plugging $\lambda = -2$ into (2), it becomes

$$\left(\begin{array}{ccc|c} 1 & -2 & -2 & 1 \\ 0 & 6 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

which solution is $(z, -\frac{1}{2}z - \frac{1}{2}, z)$, with $z \in \mathbb{R}$.

$\boxed{2}$

Consider the matrix with parameters a and b

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & b & a \\ 0 & 0 & b \end{pmatrix}.$$

(a) (5 points) For what values of the parameters a and b is the matrix A diagonalizable? Justify your answer.

(b) (5 points) For the values $a = 0$ and $b = 2$, find the matrix P and the diagonal matrix D associated to A.

Solution:

(a) Since the matrix is triangular, the eigenvalues are the elements of the diagonal, 1 and b. Note that if $b = 1$, then 1 is a triple eigenvalue. Since the matrix A is not already diagonal, it will be not diagonalizable. Suppose that $b \neq 1$. Then 1 is simple and b is a double eigenvalue. We need only to compute the rank of $A - bI = \begin{pmatrix} 1-b & 2 & 1 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$. The rank of this matrix is 2 if $a \neq 0$, and 1 if $a = 0$. Thus, A is diagonalizable if and only if $b \neq 1$ and $a = 0$.

(b) Let $b \neq 1$ and $a = 0$.

$S(1) = \langle (1, 0, 0) \rangle$

$S(b) = \langle (\frac{1}{b-1}, 0, 1), (\frac{2}{b-1}, 1, 0) \rangle$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & \frac{1}{b-1} & \frac{2}{b-1} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The case $b = 2$ is contained in the above.

$\boxed{3}$

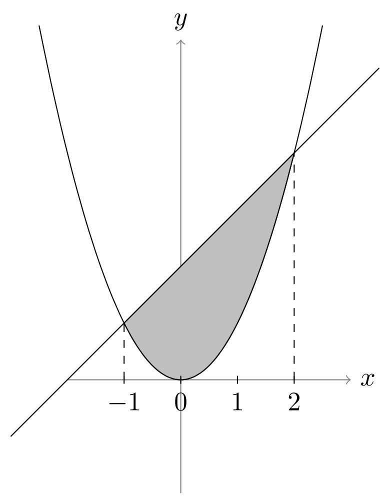
(a) (3 points) Represent the set $B = \{(x, y) \in \mathbb{R}^2 : x^2 \leq y \leq x + 2\}$. Calculate the minimum and maximum value of x if $(x, y) \in B$.

(b) (7 points) Calculate the double integral

$$\iint_B (-1 + y) dx dy.$$

Solution:

(a) $-1 \leq x \leq 2$. The region is shown below.



(b)

$$\begin{aligned}\iint_B (-1 + y) dx dy &= \int_{-1}^{2} dx \int_{x^2}^{x+2} (-1 + y) dy = \int_{-1}^{2} \left(-y + \frac{y^2}{2}\right)\Bigg|_{x^2}^{x+2} dx \\ &= \int_{-1}^{2} \left(x^2 - x - 2 + \frac{1}{2}(x + 2)^2 - \frac{1}{2}x^4\right) dx \\ &= \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{6}(x + 2)^3 - \frac{1}{10}x^5\Bigg|_{-1}^{2} = \frac{27}{10}.\end{aligned}$$

$\boxed{4}$

(a) (4 points) Given $a > 0$, calculate

$$\int_0^a x\sqrt{a^2 - x^2}dx.$$

(b) (6 points) Given the function $f(x) = \dfrac{2}{x^2 - 4x + 3}$, calculate its vertical asymptotes. Study whether the following improper integral converges

$$\int_2^4 \frac{2}{x^2 - 4x + 3}\,dx.$$

Solution:

(a) A primitive of $x\sqrt{a^2 - x^2}$ is $-\frac{1}{3}(a^2 - x^2)^{\frac{3}{2}}$. Then

$$\int_0^a x\sqrt{a^2 - x^2}dx = -\frac{1}{3}(a^2 - x^2)^{\frac{3}{2}}\Big|_0^a = \frac{1}{3}a^3.$$

(b) The denominator vanishes at the points $x = 1$ and $x = 3$. Hence, the function

$$\frac{1}{x^2 - 4x + 3}$$

is not bounded in the interval $[2, 4]$. We study the convergence of the integrals

$$\int_2^3 \frac{2}{x^2 - 4x + 3}\,dx \quad \text{and} \quad \int_3^4 \frac{2}{x^2 - 4x + 3}\,dx$$

Let $2 < b < 3$. Using partial fractions we have that

$$\begin{aligned} \int_2^b \frac{2}{x^2 - 4x + 3}\,dx &= \int_2^b \left(\frac{1}{x-3} - \frac{1}{x-1}\right)\,dx = \ln(3-x)|_2^b - \ln(x-1)|_2^b \\ &= \ln(3-b) - \ln(b-1) \end{aligned}$$

Since

$$\lim_{b \to 3^-} \ln(3-b) = -\infty$$

The integral does not converge.

A similar computation could be done by choosing $3 < b < 4$ and considering the integral

$$\int_3^4 \frac{2}{x^2 - 4x + 3}\,dx.$$

$\boxed{5}$

(a) (5 points) The sequence $\{x_n\}_{n=1}^{\infty}$ satisfies $x_{n+1} = \frac{1}{2}x_n^2 + \frac{1}{4}\sqrt{n}$, for all $n = 1, 2, \dots$ and $x_1 = 1$. Study if the sequence is convergent.

(b) (5 points) Calculate $\lim\limits_{n\to\infty} \left(\dfrac{n}{n+2}\right)^{2n}$.

Solution:

(a) It is not convergent, since it is not bounded. For, suppose that there is M such that $x_n \le M$ for all n (note that $x_n > 0$). Then, from the recurrence relation satisfied by the sequence, we have

$$\frac{1}{2}x_n^2 + \frac{1}{4}\sqrt{n} \le M,$$

thus $\sqrt{n} \le 4M - 2x_n^2 \le 4M$ for all n, hence $n \le 16M^2$ for all n, which is absurd.

Of course, other ways of proof are possible.

(b)

$$\lim_{n\to\infty} \left(\frac{n}{n+2}\right)^{2n} = \lim_{n\to\infty} \left(1 - \frac{2}{n+2}\right)^{2n} = \lim_{n\to\infty} \left(\left(1 - \frac{2}{n+2}\right)^{-\frac{n+2}{2}}\right)^{-\frac{4n}{n+2}} = e^{-4}.$$

Another possibility is to write $\left(\frac{n}{n+2}\right)^{2n} = e^{\ln\left(\frac{n}{n+2}\right)^{2n}}$ and to calculate the limit

$$\begin{aligned}\lim_{n\to\infty} \ln\left(\frac{n}{n+2}\right)^{2n} &= \lim_{n\to\infty} 2n\ln\left(\frac{n}{n+2}\right)\\ &= \lim_{n\to\infty} 2n\left(\frac{n}{n+2} - 1\right) = -4,\end{aligned}$$

where the equivalence $\ln x \approx x - 1$ when $x \approx 1$ has been used. Instead of this equivalence, we could proceed as follows: let the continuous variable x replace the discrete variable n, and calculate, by l'Hôpital Rule:

$$\begin{aligned}\lim_{x\to\infty} 2x\ln\left(\frac{x}{x+2}\right) &= \lim_{x\to\infty} \frac{\ln\left(\frac{x}{x+2}\right)}{\frac{1}{2x}}\\ &= \lim_{x\to\infty} \frac{\frac{x+2-x}{(x+2)^2}}{-\frac{1}{2x^2}}\\ &= \lim_{x\to\infty} \frac{-4x^2}{(x+2)^2} = -4.\end{aligned}$$

$\boxed{6}$

Let the series $\sum_{n=1}^{\infty} a_n$, where either $a_n = \frac{1}{2^n}$ or $a_n = \frac{1}{3^n}$, depending on the index n.

(a) (3 points) Prove that the series is convergent.

(b) (3 points) Calculate the biggest A and the smallest B such that

$$A \leq \sum_{n=1}^{\infty} a_n \leq B.$$

(c) (4 points) Let us consider now the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$, where either $a_n = \frac{1}{2^n}$ or $a_n = \frac{1}{3^n}$, depending on the index n. Calculate the smallest C such that

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n \leq C.$$

Solution:

(a) The series is positive and converges by the comparison principle, since $a_n \leq \frac{1}{2^n}$and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series of ratio $\frac{1}{2}$, hence it is convergent.

(b) The best we can do with the information provided is to state the bounds

$$\sum_{n=1}^{\infty} \frac{1}{3^n} \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} \frac{1}{2^n},$$

that is

$$\frac{\frac{1}{3}}{1-\frac{1}{3}} = 0.5 \leq \sum_{n=1}^{\infty} a_n \leq 1 = \frac{\frac{1}{2}}{1-\frac{1}{2}}.$$

(c) The series is $a_1 - a_2 + a_3 - a_4 + \cdots$. The biggest possible sum of positive terms is $\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{2}{3}$.

The smallest possible sum of negative terms is $-(\frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \cdots) = -\frac{\frac{1}{3^2}}{1-\frac{1}{9}} = -\frac{1}{8}$. Thus the sharper upper bound C of the series is

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n \leq \frac{2}{3} - \frac{1}{8} = \frac{13}{24} = C.$$