1

Consider the linear system with parameter λ.

$$\begin{cases} x + y + \lambda z &= \frac{1}{\lambda} \\ x + \lambda y + z &= \lambda \\ \lambda x + y + z &= 1 \end{cases}$$

(a) (7 points) Discuss the system.

(b) (3 points) Solve the system for the value $\lambda = 2$.

Solution:

(a) We will find an equivalent system in echelon form by applying the method of Gauss. We denote by A the matrix of the system and by A^* the augmented matrix. The system is equivalent to

$$\left(\begin{array}{ccc|c} 1 & 1 & \lambda & \frac{1}{\lambda} \\ 0 & \lambda - 1 & 1 - \lambda & \lambda - \frac{1}{\lambda} \\ 0 & 1 - \lambda & 1 - \lambda^2 & 0 \end{array}\right),$$

after the operations $\text{row}_2 - \text{row}_1$, $\text{row}_3 - \lambda\,\text{row}_1$. Now adding the second to the third row we obtain

$$\left(\begin{array}{ccc|c} 1 & 1 & \lambda & \frac{1}{\lambda} \\ 0 & \lambda - 1 & 1 - \lambda & \lambda - \frac{1}{\lambda} \\ 0 & 0 & 2 - \lambda - \lambda^2 & \lambda - \frac{1}{\lambda} \end{array}\right) \tag{1}$$

Note that $2 - \lambda - \lambda^2 = (1 - \lambda)(2 + \lambda)$.

- [Case 1] $\lambda \neq 1$ and $\lambda \neq -2$. Then $|A| \neq 0$, hence the system admits only one solution.
- [Case 2] $\lambda = 1$. Then (1) becomes
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$
Hence, the rank of A and A^* is 1, thus the system has an infinite number of solutions.
- [Case 3] $\lambda = -2$. Then (1) becomes
$$\left(\begin{array}{ccc|c} 1 & \lambda & -2 & 1 \\ 0 & -3 & 3 & -\frac{3}{2} \\ 0 & 0 & 0 & -\frac{3}{2} \end{array}\right). \tag{2}$$
Hence, A has rank 2 and A^* has rank 3, thus there is no solution.

(b) $\lambda = 2$ corresponds to [Case 1] above. Plugging $\lambda = 2$ into (1), it becomes

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & \frac{1}{2} \\ 0 & 1 & -1 & \frac{3}{2} \\ 0 & 0 & -4 & \frac{3}{2} \end{array}\right),$$

which is a echelon system which can be solved from bottom to top, to get the solution $(\frac{1}{8}, \frac{9}{8}, -\frac{3}{8})$.

2

Consider the matrix with parameters a and b

$$A = \begin{pmatrix} 4 & 2 & b \\ 1 & 3 & 2 \\ 0 & 0 & a \end{pmatrix}.$$

(a) (5 points) For what values of the parameters a and b is the matrix A diagonalizable? Justify your answer.

(b) (5 points) For the values $a = 0$ and $b = 0$, find the matrix P and the diagonal matrix D associated to A.

Solution:

(a) Expanding the determinant $|A - \lambda I|$ by the bottom row, we have

$$\begin{vmatrix} 4-\lambda & 2 & b \\ 1 & 3-\lambda & 2 \\ 0 & 0 & a-\lambda \end{vmatrix} = (a-\lambda)\big((4-\lambda)(3-\lambda)-2\big) = (a-\lambda)(\lambda^2 - 7\lambda + 10) = (a-\lambda)(\lambda-5)(\lambda-2).$$

- [Case 1] $a \neq 5$ and $a \neq 2$. A has 3 different proper values, hence A is diagonalizable for all b.
- [Case 2] $a = 5$. A has the simple proper value 2 and the double 5. We only need to calculate the rank of $A - 5I = \begin{pmatrix} -1 & 2 & b \\ 1 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, which is 1 if and only if $b = -2$.
- [Case 3] $a = 2$. A has the simple proper value 5 and the double 2. We only need to calculate the rank of $A - 2I = \begin{pmatrix} 2 & 2 & b \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} I$, which is 1 if and only if $b = 4$.

In conclusion, A is diagonalizable if and only if ($a \neq 5$ and $a \neq 2$, any b) or ($a = 5$ and $b = -2$) or ($a = 2$ and $b = 4$).

(b) Let $a = b = 0$. The eigenvalues are 0, 5, 2.

$S(0) = \langle (2, -4, 5) \rangle$

$S(5) = \langle (2, 1, 0) \rangle$

$S(2) = \langle (1, -1, 0) \rangle$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad P = \begin{pmatrix} 2 & 2 & 1 \\ -4 & 1 & -1 \\ 5 & 0 & 0 \end{pmatrix}.$$

3

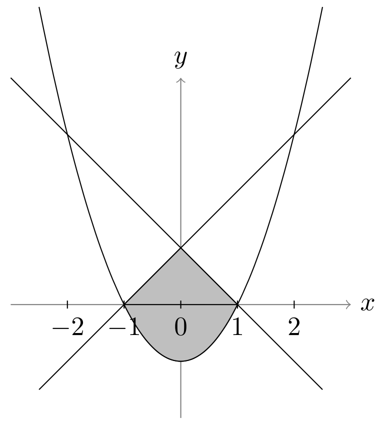
(a) (3 points) Represent the set $B = \{(x,y) \in \mathbb{R}^2 \,:\, y \leq x+1, y \leq 1-x, y \geq x^2 - 1\}$.

(b) (7 points) Calculate the double integral

$$\iint_B x^2 dxdy.$$

Solution:

(a) The set B is the shadow region in the figure below.



(b) It is needed to split the integral. This can be done in different ways, but the most direct way is

$$\iint_B x^2 dxdy = \int_{-1}^{0} x^2 dx \int_{x^2-1}^{x+1} dy + \int_0^1 x^2 dx \int_{x^2-1}^{1-x} dy$$

The first integral is

$$\int_{-1}^{0} x^2 dx \int_{x^2-1}^{x+1} dy = \int_{-1}^{0} x^2(x+1-x^2+1)dx = \int_{-1}^{0} (2x^2+x^3-x^4)dx = \frac{2}{3} - \frac{1}{4} - \frac{1}{5}.$$

The second integral is

$$\int_0^1 x^2 dx \int_{x^2-1}^{1-x} dy = \int_0^1 x^2(1-x-x^2+1)dx = \int_0^1 (2x^2-x^3-x^4)dx = \frac{2}{3} - \frac{1}{4} - \frac{1}{5}.$$

Adding both results we get the final answer: $\frac{13}{30}$.

4

(a) (6 points) Given $a > 0$, classify the quadratic form $Q(x, y, z) = ax^2 + 4ay^2 + 4az^2 + 4xy$.

(b) (4 points) Study whether the following improper integral converges

$$\int_4^\infty \frac{2}{x^2 - 4x + 3}\, dx.$$

If it converges, compute its value

Solution:

(a) The principal minors are $D_1 = a$, $D_2 = 4a^2 - 4$, $D_3 = 4a(4a^2 - 4)$.

- $[Q\ D^+]$ $D_1 > 0$, $D_2 > 0$, $D_3 > 0$ iff $a > 0$ and $|a| > 1$, that is, $a > 1$;
- $[Q\ SD^+]$ $D_1 > 0$, $D_2 = 0$, $D_3 = 0$ iff $a = 1$;
- $[Q\ D^-]$ $D_1 < 0$, $D_2 > 0$, $D_3 < 0$ iff $a < 0$ and $|a| > 1$, that is, $a < -1$;
- $[Q\ SD^-]$ $D_1 < 0$, $D_2 = 0$, $D_3 = 0$ iff $a = -1$;
- $[Q\ I]$ for the rest of values of a, that is, $-1 < a < 1$.

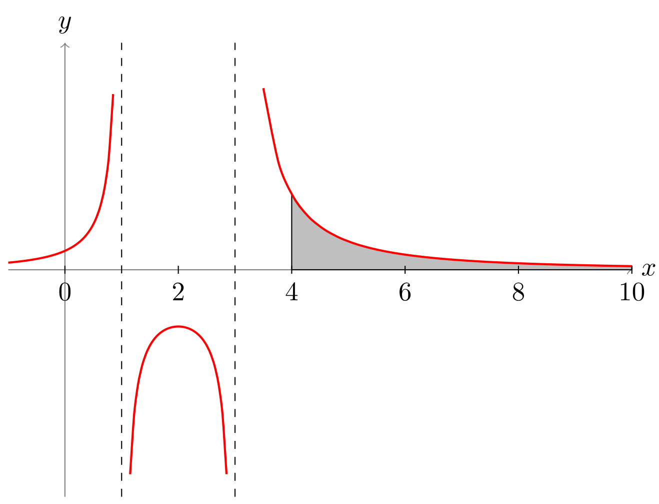
(b) Let $4 < b < \infty$. Using partial fractions we have that

$$\begin{aligned}\int_4^b \frac{2}{x^2 - 4x + 3}\, dx &= \int_4^b \left(\frac{1}{x-3} - \frac{1}{x-1}\right) dx = \ln(x-3)|_4^b - \ln(x-1)|_4^b \\ &= \ln(b-3) - \ln(4-3) - \ln(b-1) + \ln(4-1) \\ &= \ln\left(\frac{b-3}{b-1}\right) + \ln 3.\end{aligned}$$

Since

$$\lim_{b\to\infty} \ln\left(\frac{b-3}{b-1}\right) = \ln\left(\lim_{b\to\infty}\frac{b-3}{b-1}\right) = \ln 1 = 0,$$

the integral is convergent and its value is $\ln 3$. The picture below shows the graph of f. The value of the integral is the area of the (infinite) shadow region.



5

Given the sequence $x_{n+1} = x_n^2 + \frac{1}{n}$, for all $n = 1, 2, \ldots$ and $x_1 = 1$, answer the following questions.

(a) (5 points) Study whether the sequence is monotone.

(b) (5 points) Study whether the sequence is convergent, and in this case calculate its limit as $n \to \infty$.

Solution:

(a) $x_2 = 2 > 1 = x_1$. Now, assuming that $x_n > 1$ when $n > 1$, we have

$$x_{n+1} = x_n^2 + \frac{1}{n} > 1, \quad x_{n+1} > x_n^2 > x_n, \text{ since } x_n > 1.$$

(b) Since $x_2 = 2, x_3 > 2^2, x_4 > x_3^2 > 2^{2\cdot 2}$ and hence recursively $x_{n+2} > 2^{2\overset{n)}{\cdots}2} = 2^{2^n} \longrightarrow \infty$. Hence $x_n \longrightarrow \infty$.

6

Let the series $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{n}{n^2+1}$.

(a) (5 points) Study if the series is absolutely convergent.

(b) (5 points) Study if the series is convergent and in this case, give an estimate of the maximum error committed when the infinite series is approximated by the sum of the first three terms.

Solution:

(a) Note that taking the general term in absolute value we obtain $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{n}{n^2+1}\right| = \sum_{n=1}^{\infty}\frac{n}{n^2+1}$, which has the same character than the harmonic series, thus it diverges (note that $\frac{n}{n^2+1} \geq \frac{1}{2}\frac{1}{n}$, since $2n^2 \geq n^2+1$ for all $n \geq 1$). Thus, the series is not absolutely convergent.

(b) The series is an alternate series. Let us check whether the conditions of Leibniz's Theorem are fulfilled, so it is convergent. Moreover, we can obtain the estimates asked for in the exercise.

- The general term is monotone decreasing, since $\frac{n}{n^2+1} > \frac{n+1}{(n+1)^2+1} \Leftrightarrow n\big((n+1)^2+1\big) > (n+1)(n^2+1)$, due to $n^3+2n^2+n+1 > n^3+n^2+n+1$.
- The general term tends to 0.

Hence, the series is convergent and calling S its sum and $S_3 = \frac{1}{2} - \frac{2}{5} + \frac{3}{10} = \frac{4}{10} = 0.4$, we have

$$|S - S_3| < \frac{4}{4^2+1} = a_4.$$

Hence, the infinite sum is 0.4, with error less than $\frac{4}{17} \approx 0.2941$.